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Motive, Collection, and Voice Leading in John Coltrane's "Giant Steps"

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ABSTRACT
Coltrane's "Giant Steps" is a tightly-woven, theoretically-dense composition containing zero-sum characteristics, hexatonic and nonatonic properties, and various types of patterns on both the surface and deeper levels of the piece. However, the theoretical neatness of "Giant Steps" is offset by several irregularities, creating a cohesive and nuanced whole. This article examines "Giant Steps" from several different analytical perspectives, and considers mathematically the question of whether certain nonatonic aspects of the piece could have occurred by chance or whether they were necessarily part of Coltrane's compositional design.

Introduction

This article explores motive, collection, and voice-leading in John Coltrane's seminal composition "Giant Steps" (GS).[1] Part I of the essay focuses on the collections used in GS. The first section of Part I establishes the relationships between the collections themselves – particularly the hexatonic and nonatonic – information that will aid in understanding the remainder of the article. Coltrane's possible derivation of a portion of GS from Slonimsky is then considered.[2] This is followed by a section investigating mathematically the probability that the nonatonicism of GS could have resulted by chance rather than by compositional design. (Two appendices address mathematical aspects of this question in greater detail.) Part II of the essay explores voice-leading, focusing on the zero-sum properties of GS by realizing its progressions using different chord voicings. Part III is concerned with motivic aspects of GS, and presents my composition "Dual Duel" as a foil.

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[1]"Giant Steps" (composed by John Coltrane, Copyright © 1974 (Renewed 2002), JOWCOL MUSIC LLC. This arrangement Copyright © 2019 JOWCOL MUSIC LLC. International Copyright Secured. All Rights Reserved.) appears on *Giant Steps*, Atlantic 1311 (1960, recorded 1959). Excerpts are reprinted by permission of Hal Leonard LLC. I would like to thank Daniel Shanahan, Andrew Aziz, Lewis Porter, and especially Keith Waters for their comments on the manuscript overall. I would also like to thank Holly Dinkel at NASA, John Pellegrin at Motorola, and especially my statistics colleague at the University of Florida, Rohit Patra, for their assistance with the section of the paper involving mathematical probabilities.

[2]Nicholas Slonimsky, *Thesaurus of Scales and Melodic Patterns* (New York: Scribner, 1947).

Figure 1. Two nonatonic scales found in work of Demsey (above) and Santa (below). Demsey, "Chromatic Third Relations," 172–73; Santa, "Nonatonic Progressions," 13, 16.

Part I: Collection

Hexatonic and Nonatonic Relationships

Figure 1 shows two nine-note scales. The first appears in David Demsey's work, which builds upon that of Andrew Jaffe.[3] Remarkably, this scale is identical to both the complete set of melodic tones in GS and the complete set of chord roots in GS, a fact I will investigate further in the section below, "Nonatonic Collections and the Probability of Coincidence."[4] The second scale in Figure 1 corresponds to the Western nonatonic system found in Matthew Santa's article, "Nonatonic Progressions in the Music of John Coltrane."[5] I have added a beam connecting the notes that are tonal centers in GS to make the diagram more similar to Demsey's.[6]

The difference between the two scales at first appears to simply be that the second note of each three-note segment is a half-step lower in Santa's scale. However, when the intervallic sequence of each major-third segment is considered – 211 for Demsey and 121 for Santa – it can be seen that the intervals are rotated. (The only way, in fact, that a major third can be divided into three parts – assuming division of the octave into twelve semitones – is with two semitones and one whole tone.) The two scales are therefore transposed rotations of one another, and may be referred to as "nonatonic collections."[7]

[3]David Demsey, "Chromatic Third Relations in the Music of John Coltrane," *Annual Review of Jazz Studies* 5 (1991): 172–3; Andrew Jaffe, *Jazz Harmony* (Dubuque, IA: William C. Brown, 1983), 170.

[4]Demsey only mentions that these are the melodic tones of the second half of GS, but the first half uses a subset of this scale.

[5]Matthew Santa, "Nonatonic Progressions," *Annual Review of Jazz Studies* 13 (2003): 13, 16. Nonatonic systems are explained in detail in Part II of the present article. Santa's Western nonatonic system is shown in Figure 11.

[6]In the beginning of the article Santa presents a nonatonic scale and then relates it to a nonatonic system (see pages 13–14). Subsequently, he never specifically lays out the scales corresponding to each nonatonic system, probably because the notes of each system form a collection with no particular tonal center. However, in the example he does give, Santa uses the root of the chord at the top of the system (C major in this case) to begin the scale, and I have followed suit. The fact that Santa's Western system begins with E♭ (like Demsey's) is simply a result of the fact that he begins with C major at the top of the Northern system, and then proceeds in clockwise fashion, placing D♭ major at the top of the Eastern system, D major at the top of the Southern system, and E♭ major at the top of the Western system (16). Santa's nonatonic systems are an extension of Richard Cohn's hexatonic systems, which are presented similarly. Richard Cohn, "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions," *Music Analysis* 15, no. 1 (1996): 9–40.

[7]Nonatonic collections are equivalent to the third of Messiaen's modes of limited transposition. Olivier Messiaen, *Technique de mon langage musical*, vol. 1 (Paris: Alphonse Leduc, 1944), 58–63.

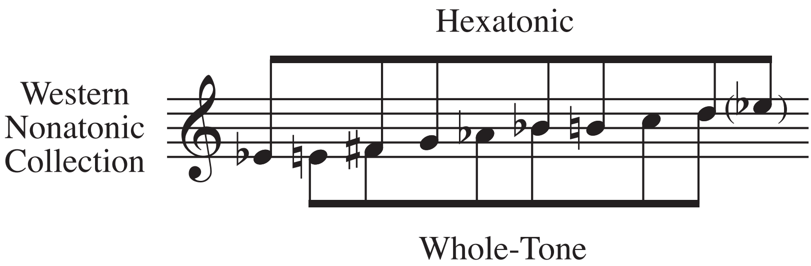

Figure 2. The Western nonatonic collection as a combination of two overlapping hexachords – the hexatonic and whole-tone. The overlapping tones form an augmented triad.

A nonatonic collection can be conceived of as the combination of two overlapping hexachords – one we commonly call "hexatonic" and one we commonly call "whole-tone" – both of which are themselves symmetrical (see Figure 2). The shared notes form an augmented triad. A nonatonic collection may also be created by combining two hexatonic collections. The relationships between nonatonic collections, hexatonic collections, and their greatest common chordal factor, the augmented triad, are illustrated in Figure 3. The top portion of the diagram presents the four nonatonic collections as scales along with their geographic names.[8] In the bottom half of the diagram, nonatonic collections are identified with subscripts abbreviating their geographic names; e.g. Non_W indicates the Western nonatonic collection. Lines indicate motion in either direction, as, for example, the hexatonic collections both contain augmented triads and combine to create nonatonic collections. Just as augmented triads act as pivots between different hexatonic collections, so do hexatonic collections act as pivots between different nonatonic collections.

This pivot structure may also be seen in diagrams created by other scholars, if their meaning is somewhat adapted. Cohn illustrates the relationship between hexatonic collections and Weitzmann regions with a diagram showing "four Weitzmann water bugs in union with four hexatonic pools."[9] Each water bug in the diagram straddles two hexatonic pools. A Weitzmann region is a specific set of major and minor triads, not a pitch-class collection – such as a particular hexatonic or nonatonic collection – that may be arranged in different ways. However, the pitch-class content of each of the four Weitzmann regions is in fact equivalent to that of the four nonatonic collections. A Weitzmann region may therefore be considered as a specific type of nonatonic system – one generated by an ***N***/***R*** chain – just as other types of nonatonic systems are presented in Santa's work and in this article.[10] Each water bug can then be

[8]As with Cohn's hexatonic systems, the geographic names for nonatonic systems are used "for heuristic purposes." Matthew Santa, "Nonatonic Systems and the Interpretation of Dominant/Tonic Progressions," *Theory and Practice* 28 (2003): 3.

[9]Richard Cohn, *Audacious Euphony: Chromatic Harmony and the Triad's Second Nature* (New York: Oxford University Press, 2012), 85. For more on Weitzmann regions, see ibid., chapters four and five.

[10]An ***N***/***R*** chain consists of an alternation between the *Nebenverwandt* (***N***) transformation and the *Relative* (***R***) transformation. The ***N*** transformation maps a major triad to its minor subdominant or a minor triad to its major

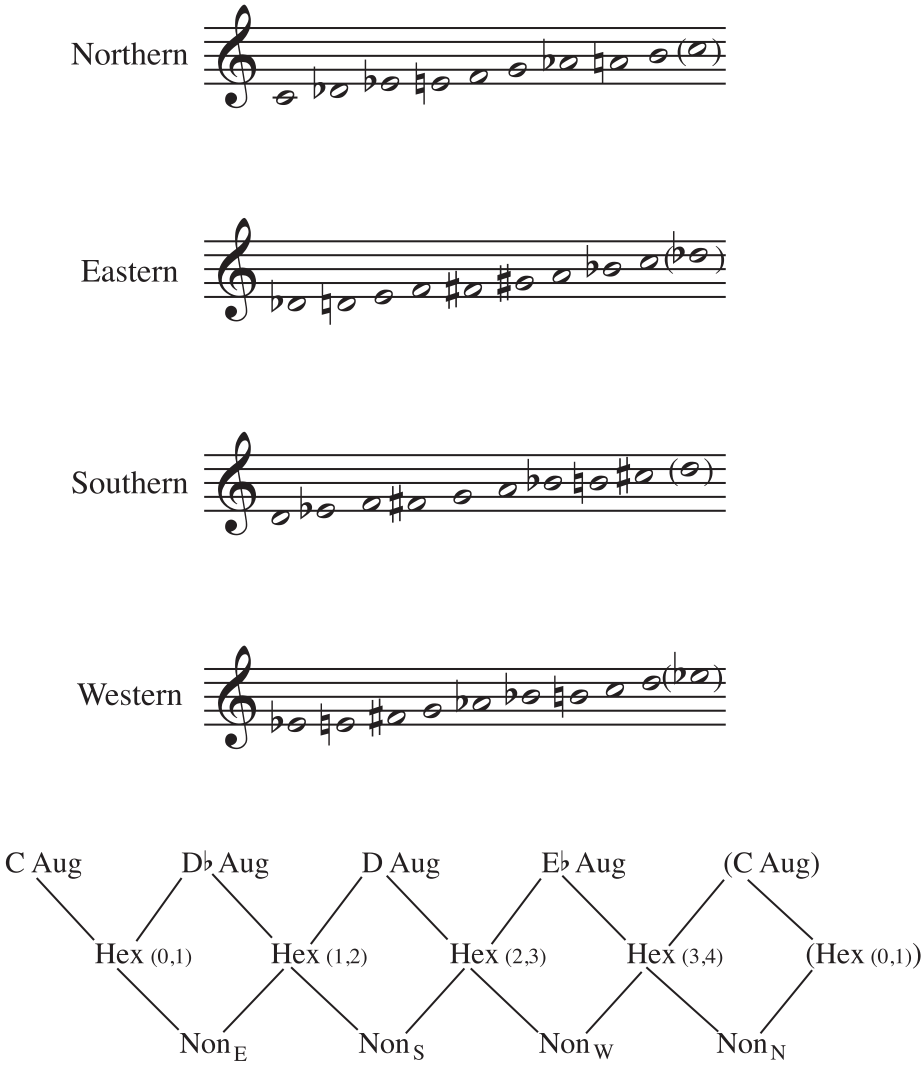

Figure 3. The four nonatonic collections (Non) and their relationships with augmented triads (Aug) and hexatonic collections (Hex).

envisioned as the union of two adjacent hexatonic collections, rather than as triadic subsets of each, and it becomes apparent that hexatonic collections act as pivots between nonatonic collections. This pivot structure may also be seen in Jack Douthett and Peter Steinbach's "Cube Dance" diagram, which is superimposed in Cohn's voice-leading zones diagram.[11]

dominant; e.g., C major becomes F minor, and vice versa. The ***R*** transformation maps a triad to its relative major or minor triad; e.g., C major becomes A minor, and vice versa.

[11] Jack Douthett and Peter Steinbach, "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition," in "Neo-Riemannian Theory," special issue, *Journal of Music Theory* 42, no. 2 (1998): 254; Cohn, *Audacious Euphony*, 104. The numbers 0–11 in Cohn's diagram are useful for understanding the pivot structure, but must be regarded merely as references to locations in "Cube Dance," not as voice-leading zone designations. If considered in this way, it can be seen that the pitch-class content of the chords in "Cube

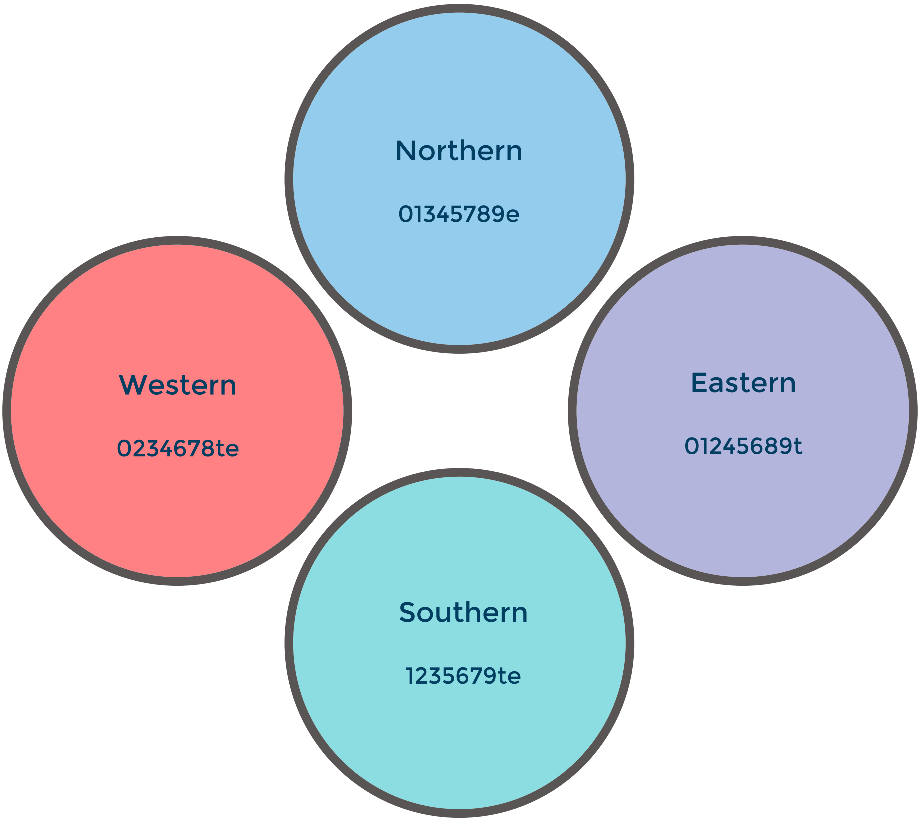

Figure 4. The four nonatonic collections and their pitch-class content, shown as discrete, unordered sets.

These observations regarding hexatonic and nonatonic relationships are nontrivial. For example, in his article applying nonatonic systems to nineteenth-century music, Santa writes that "hexatonic and nonatonic systems that share the same group of major triads will be defined here as *parallel* systems."[12] He then explains the "marriage of parallel hexatonic and nonatonic systems" in "hybrid hexatonic/nonatonic systems," which are again arranged into Northern, Eastern, Southern, and Western systems.[13] However, as we have seen, any two nonatonic collections share a hexatonic collection, and thus a group of three major triads. Therefore, "marriages" are possible between *other* hexatonic and nonatonic systems that also share the same group of major triads.[14]

Dance" at locations 1 and 2 are each equivalent, both separately and jointly, to $Hex_{(0,1)}$; locations 4 and 5 are equivalent to $Hex_{(1,2)}$. These four locations (1, 2, 4, and 5) together are equivalent to Non_E, as are locations 0–6. Similarly, locations 3–9 are equivalent to Non_S, 6–0 are equivalent to Non_W, and 9–3 are equivalent to Non_N, with each nonatonic collection containing its own internal hexatonic structure, as described for Non_E.

[12]Santa, "Nonatonic Systems," 11 (emphasis original).

[13]Ibid., 11–14.

[14]For example, Santa's Western hybrid system uses the pitch classes of the Western nonatonic collection, which includes the E♭, G, and B major triads from the Western hexatonic collection. (Geographic names for hexatonic collections are used below, in Figure 8.) But the Southern nonatonic collection also contains the E♭, G, and B major triads from the Western hexatonic collection. A "Southwestern" hybrid system would therefore also be

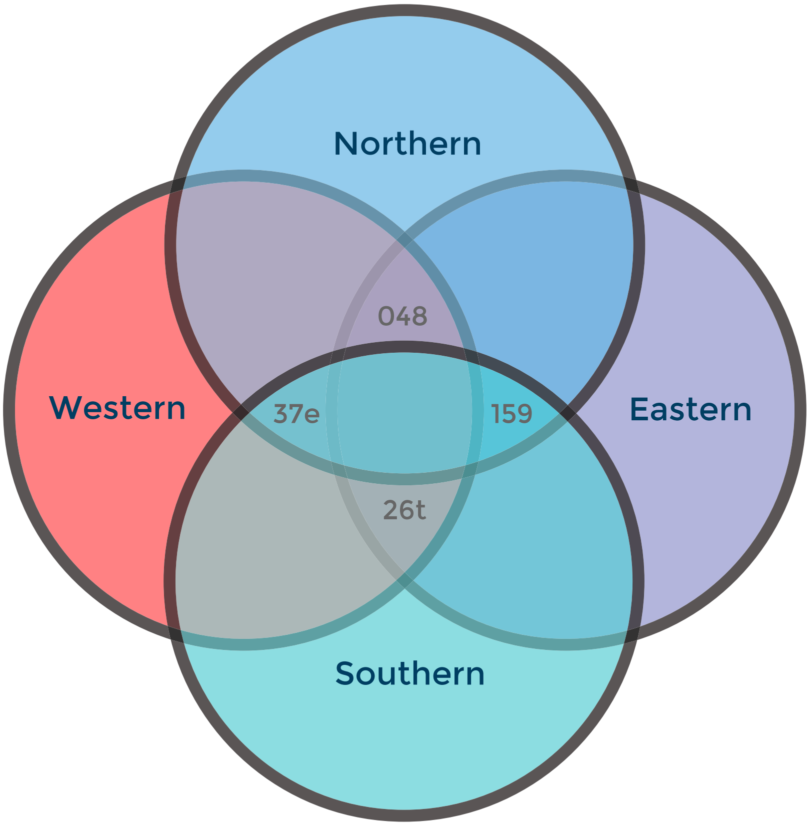

Figure 5. The four nonatonic collections and their pitch-class content, shown as intersecting, unordered sets.

I will now examine the relationships between augmented triads, hexatonic collections, nonatonic collections, and whole-tone collections from a different perspective, facilitating further understanding of them and laying necessary groundwork for the discussion of probability occurring later in Part I. Figure 4 shows the four nonatonic collections and their pitch-class content as discrete, unordered sets.[15]

Figure 5 presents the same information as intersecting, unordered sets. In this presentation, we can see how *one* nonatonic collection is comprised of *three* out of four of the possible augmented triads. Conversely, any group of *three* nonatonic collections contains three intersecting pitch classes in the form of *one* augmented triad. Any group of *two* nonatonic collections contains six intersecting pitch classes, in the form of *two* augmented triads. (The region at the center of the diagram is empty because no pitch classes

possible. (However, a Southwestern system would require a different Southern nonatonic system to be created out of the pitches classes of the Southern nonatonic collection, as Santa's Southern system does not use the E♭, G, and B major triads.)

[15]Curly brackets denoting unordered sets are not used in these diagrams but are implied by the boundaries of the regions in which integers appear.

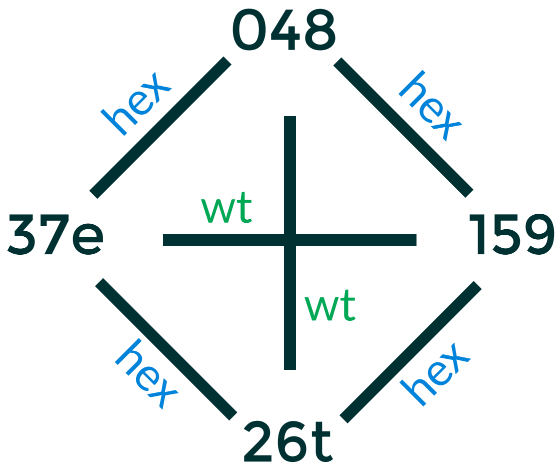

Figure 6. Augmented triads extracted from Figure 5 and their relationship to the four hexatonic and two whole-tone collections.

are common to all four nonatonic collections.)[16] For geographically adjacent collections (e.g. Northern and Eastern), these two augmented triads form a hexatonic collection. This is shown in Figure 6, which extracts the four augmented triads from Figure 5.

For non-adjacent collections (e.g. Northern and Southern), these two augmented triads form a whole-tone collection. Figure 7 superimposes Figure 6 back onto Figure 5 using generic designations to illustrate the relationships between all of these elements. Figure 8 shows the same information using specific designations for each chord and collection.

The above observations regarding hexatonic and nonatonic collections are significant for my examination of GS, as will be seen throughout much of this article. Parsing GS in various ways produces either the Western or Southern nonatonic collection. Together, these two collections form the aggregate (as do any two nonatonic collections), reflecting the highly chromatic nature of GS.[17] By contrast, if we consider what they have in common, we find that at the center of GS lies a hexatonic core – $\text{Hex}_{(2,3)}$ – which will be further explored in the section of Part II titled "A Hexatonic Realization."

"Giant Steps" and the Slonimsky Thesaurus

Demsey, with Robert Wason's assistance, demonstrates that an example from the introduction of Slonimsky's *Thesaurus* – a harmonization of his pattern 646 – was likely used as source material for the second half of GS, and also

[16]This information about the number of intersecting pitch classes in different combinations of nonatonic collections will be used directly in Appendix 1.

[17]The missing three notes from each nonatonic collection form an augmented triad. Regarding the chromaticism of GS in general, see the pitch-class circulation diagrams of GS appearing in Dmitri Tymoczko, *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice* (New York: Oxford University Press, 2011), 164–7.

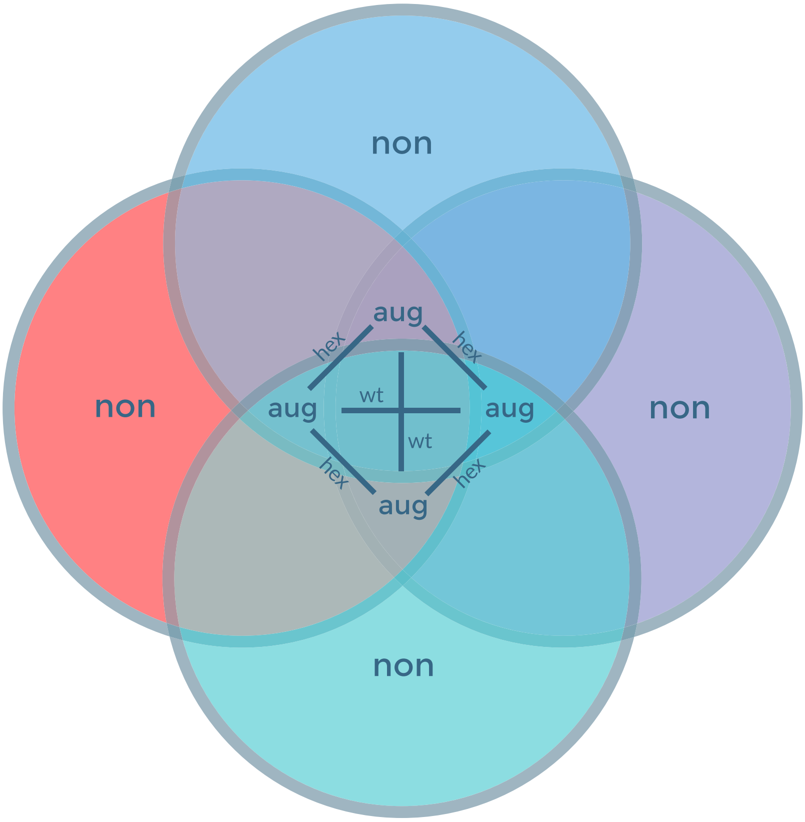

Figure 7. Figure 6 superimposed back onto Figure 5 – both with generic designations – to show the relationships between augmented triads, whole-tone collections, hexatonic collections, and nonatonic collections.

points to Slonimsky's pattern 286 (see Figures 9 and 10).[18] Demsey's discussion is unquestionably significant, but I would like to put some of his findings into perspective before proceeding.

To begin with, there are several differences between the second half of GS and the example in Slonimsky's introduction (Figure 9). (See Figure 25 for a simplified lead sheet of GS.) Slonimsky's examples are presented as continuous streams of a single note-value, and thus are essentially arrhythmic; GS, by contrast, is rhythmicized. Slonimsky's pattern contains one additional melodic tone at the end of each sequential cell.[19] This final note is approached with a descending seventh, which causes the sequence to descend rather than ascend, changing its relationship to the ascending-thirds key cycle. Different iterations of the theme (inhead/outhead, first-time/repeat) vary from one another, and Coltrane often repeats the first note of a cell rather than move down by whole step, as

[18]Demsey, "Chromatic Third Relations," 156–7; Slonimsky, *Thesaurus*, vi; ibid., 40.
[19]Sequential cells in the Slonimsky example are shown in brackets in Figure 9. The analogous cells in the second half of GS occur in two-measure units.

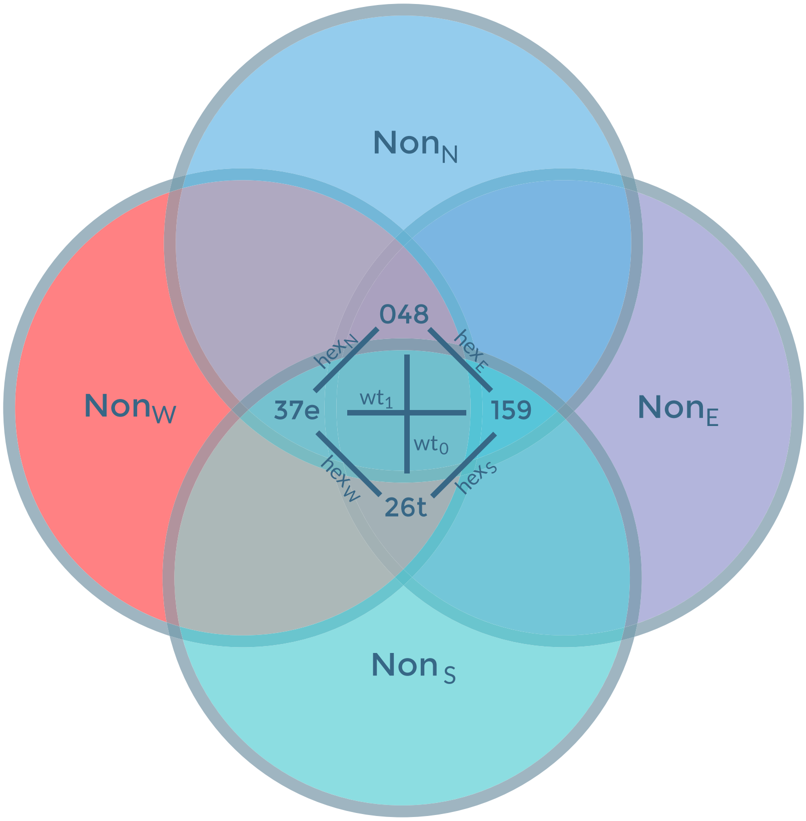

Figure 8. The relationships between specific augmented triads, whole-tone collections, hexatonic collections, and nonatonic collections.

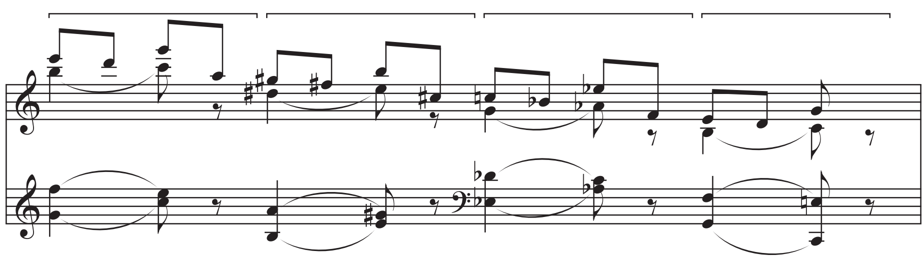

Figure 9. Pattern given in the introduction to Slonimsky's *Thesaurus* (page vi). Brackets indicate sequential cells.

Figure 10. Pattern 286 from Slonimsky's *Thesaurus* (page 40).

Slonimsky does.[20] He adds a supertonic chord to each key area. Slonimsky's melody is dodecaphonic, whereas Coltrane's omission of the last note creates a nonatonic collection that is adhered to throughout the composition in both the melody and chord roots, as discussed below.

It is true that the melodic differences could have been derived from pattern 286 (aside from the repeated-note alteration Coltrane introduces). However, as Demsey notes, Slonimsky presents "all of the possible melodic patterns" arising from each division of the octave(s).[21] Why, then, should pattern 286 be regarded as a "surprisingly direct connection" to GS?[22] We know that Coltrane practiced out of Slonimsky, but whether or not he was familiar with pattern 286 or consciously borrowed it seems not to matter. If all possible equal-subdivision patterns appear in the *Thesaurus*, then any such patterns composed after 1947 would have already been published by Slonimsky, whether or not a particular composer was aware of this fact. In addition, Slonimsky aimed to provide a comprehensive catalog of patterns specifically for composers to use as a resource: "The scales and melodic patterns in the *Thesaurus* are systemized in a manner convenient to composers in search of new materials."[23]

Nonatonic Collections and the Probability of Coincidence

Santa observes that the melody notes of GS are equivalent to the Southern nonatonic collection, and writes that "the odds of a random sampling of twenty-five pitch classes adhering solely to a single nonatonic collection are less than 3%, a

[20]For further observations regarding the different iterations of the head, see note 49 of this article; Keith Waters, "'Giant Steps' and the ic4 Legacy," *Intégral* 24 (2010), special issue in honor of Robert Wason: 135–62; and Henry Martin, "Expanding Jazz Tonality: The Compositions of John Coltrane," *Theory and Practice* 37/38 (2012–13): 185–219.

[21]Demsey, "Chromatic Third Relations," (156). 1,330 single-line patterns are given in the *Thesaurus* (in addition to several other pages of miscellaneous single-line material). Pattern 286 is one of thirteen patterns specifically applying infra-interpolation to equal division of the octave into three parts. In addition, all patterns based on infra-ultrapolation, inter-ultrapolation, and infra-inter-ultrapolation are presented. (The prefixes "infra" and "ultra" describe the location of an inserted tone – below or above, respectively – relative to the following tone. The prefix "inter" indicates that the inserted tone lies vertically between the tones which precede and follow.)

[22]Demsey, "Chromatic Third Relations," 157.

[23]Slonimsky, *Thesaurus*, i. There is also the seemingly minor issue of an erroneous caption accompanying Demsey's Figure 3 (see page 156). The caption reads, "From Slonimsky, *Thesaurus of Scales and Melodic Patterns*, 29," whereas the material presented is drawn from pages 27, 29, 34, 40, 41, 43, and 45 of the *Thesaurus*. This may have been an editorial decision, but the result is unfortunately that readers could be misled. If the material is understood to appear entirely on page 29, the implication is that the figure is taken directly from the *Thesaurus*; the caption wording "from Slonimsky" reinforces this sense. It would then be assumed that "Sample Infra-Interpolation" are Slonimsky's own words and that he specifically used this pattern to exemplify infra-interpolation; the accompanying text "'Giant Steps,' mm. 8–16" is set in a different font style and therefore would appear to be an annotation of the original Slonimsky material. Slonimsky's example also runs up and down two octaves in sixteenth notes, whereas Demsey presents the pattern ascending one octave in whole notes and quarter notes, giving it an appearance similar to GS. Close readers or those familiar with the *Thesaurus* would understand that this material could not all occur on page 29. However, with a caption font much larger than the tiny font used in the figure, some readers may simply remember seeing a page from the *Thesaurus* featuring a single, sample infra-interpolation pattern that looks very similar to the second half of GS. This is a quite different sense than one gets from surveying the 250-page *Thesaurus* itself.

fact that suggests this correspondence is significant."[24] This is significant indeed, but even more so when we recall Demsey's observation that all of the chord roots also adhere solely to this nonatonic collection.[25]

Let us first consider the melody notes alone. The probability that twenty-five pitch classes, selected randomly in independent trials, forms the Southern nonatonic collection or a subset thereof may be calculated as follows:

$$(9/12)^{25} = 0.00075254345 = 0.075\%$$

There is a nine in twelve chance for each of twenty-five trials that a pitch class from the Southern nonatonic collection will be selected, and probabilities are multiplied over successive trials.[26]

The (rounded) percentage of 0.075 is suggestive for understanding Santa's upper limit of 3%. He considered the probability of "adhering solely to a single nonatonic collection." A "single" collection, as opposed to a particular one, could potentially refer to any collection of nine pitch classes; i.e. one with no preselection criteria. The probability for an unspecified collection of nine pitch classes would be significantly higher than that for the Southern nonatonic collection, but is also far more complex to determine, so an estimation could have been made. Another possibility is that "a single nonatonic collection" refers to any of the four nonatonic collections described in the article. This is logical, because the four nonatonic collections are transpositionally equivalent. To arrive at the probability that any of the four collections are adhered to, the probability for each is added together:

$$0.075\% + 0.075\% + 0.075\% + 0.075\% = 0.3\%$$

A simple decimal error could have yielded 3% rather than 0.3%.

Because nonatonic collections are non-mutually exclusive, one would normally need to add and subtract several layers of intersecting probabilities in order to ensure that each is counted once and only once. This process is described in detail in Appendix 1, which refers to the present location in the main text. However, the approach taken below rules out the possibility of intersecting events, as is further explained in Appendix 2. One point is worth making here: considering non-mutual exclusivity results in a lower probability, as the net effect of the process is that redundancies are subtracted out. Apart from the

[24]Santa, "Nonatonic Progressions," 21. Santa, again apparently not working from Demsey, shows the Southern Nonatonic Collection here starting on F rather than E♭, as in Demsey's version. However, this is also inconsistent with Cohn and with Santa's practice earlier in the article (see note 4).

[25]Santa never mentions Demsey's scale, but cites his article in the opening annotation. Demsey also considers the question of whether the use of a nonatonic collection in GS could be coincidental, but approaches the issue from a different perspective (see pages 173–4).

[26]For a broad investigation of music and probability, see David Temperley, *Probability and Music* (Cambridge, MA: MIT Press, 2007).

decimal, a probability of "less than 3%" is thus fairly accurate and avoids the complicated issue of non-mutual exclusivity.[27]

Before further considering the probability for four nonatonic collections, I will refine our calculation of the probability for one. The percentage of 0.075 includes the possibility of a subset of the Southern nonatonic collection being formed, as noted above. To eliminate this possibility the calculation must be modified as follows:

$$(9/12)^{25} - (8/12)^{25} = 0.00071294133 = 0.071\%$$

This approximately eliminates by subtraction the probability that a subset of eight or fewer pitch classes is formed, thus yielding the probability that the Southern nonatonic collection in its entirety is formed. (NB: This calculation is a simplification. A more precise analysis is provided in Appendix 2.)

The resulting probability is very low. But when the twenty-six chord roots are considered as well, the probability is extraordinarily low, due to exponentiation. With twenty-five melody notes and twenty-six chord roots, there are a total of fifty-one pitch classes:

$$(9/12)^{51} - (8/12)^{51} = 0.00000042369 = 0.00004\%$$

This is equivalent to a 1 in 2.5 million chance.[28] Finally, this probability must be added to the probabilities of arriving at any of the other three nonatonic collections:

$$0.00004\% + 0.00004\% + 0.00004\% + 0.00004\% = 0.00016\%$$

This is less than a 1 in 600,000 chance.

However, we must now consider factors that increase the probability. There is less than a 1 in 600,000 chance of fifty-one randomly-selected pitch classes forming one of the four nonatonic collections. But composers do not typically select pitches randomly; rather, context informs their decision-making process. For example, a composer might choose to write a sequence, repeating some material at a different pitch level. The first half of GS is comprised of two iterations of a harmonic-melodic sequence that descends by major third (discussed below). This raises the odds that a symmetrical nonatonic scale will occur. At the same time, the sequence raises the question of compositional intent, a controversial matter. One may find it perfectly reasonable that Coltrane intended to compose a sequence, but have doubts that he intended to use a symmetrical nonatonic collection.

I will now further explore this issue by reconsidering the derivation from Slonimksy in this context. It seems fairly likely that Coltrane was influenced by the

[27] Regardless of the decimal, it is still a true statement that the probability is less than 3%.

[28] In probability theory, this conjures images related to the so-called infinite monkey theorem – millions of monkeys typing "random" notes.

pattern in Slonimsky's introduction. I am less convinced that he specifically used pattern 286, and find it inconsequential if he did, since that is the purpose of the *Thesaurus* and use of the pattern would still demonstrate that Coltrane was aware of this significant element of design. However, for the sake of argument, I will adjust the calculations to account for this possibility. Suppose that Coltrane consciously based the second half of GS on pattern 286. What is the probability that his use of the same nonatonic collection in the first half was merely coincidental?

$$(9/12)^{28} = 0.00031747927 = 0.032\%$$

There are twenty-eight pitch classes in the first half of GS: fourteen melody notes and fourteen chord roots. (This is true whether we consider the first half to be mm. 1–8 or mm. 16–7.) We are only concerned with a particular nonatonic collection, the Southern, rather than all four possibilities, since the same nonatonic collection is used in the first half as in the second half. I will disregard the issue of subsets as it affects the probability by only a small amount and is complicated by the fact that the first half of GS *is* actually a subset. (As discussed below, the first half is almost entirely hexatonic, but Coltrane uses a total of eight pitch classes, all drawn from the Southern nonatonic collection.)[29]

One might observe that the fourteen chord roots are drawn from the same nonatonic collection because Coltrane used I, ii, and V chords from the same three key areas that are used in the second half. Like the sequence in the first half, this three-key scheme is an element of design that raises the probability of the Southern nonatonic collection occurring throughout the piece, but also points to Coltrane's awareness of compositional structure, demonstrating the push–pull dynamic at work in such questions. In any case, we may also calculate the probability that the fourteen melody notes alone are all drawn from the same nonatonic collection, disregarding the fourteen chord roots:

$$(9/12)^{14} = 0.01781794801 = 1.78\%$$

Even if one wishes to remove seven melody notes from this calculation due to the sequence, the probability is still very low:

$$(9/12)^{7} = 0.13348388671 = 13.35\%$$

One could go further still and remove two notes from these seven, since Coltrane uses the motivic cell from the second half once in each iteration of the sequence:

$$(9/12)^{5} = 0.2373046875 = 23.73\%$$

(The third note of this melodic cell, B-A-**D**, is elided with the second iteration of the sequence, and thus has already been removed from the calculation.)

[29]The total of eight pitch classes is the same whether one considers the first half to be mm. 1–8 or mm. 16–7.

How should these findings be interpreted? First, it should be noted that 23.73% is high enough that it is possible Coltrane did use the nonatonic collection by coincidence. However, most of the considerations mentioned above revolve around the initial allowance – made only for the sake of argument – that Coltrane consciously used Slonimsky's pattern 286, and that single factor itself reduces the exponent by 23 (from 51 to 28). Furthermore, it is hard to imagine that Coltrane would choose to compose with pattern 286 but somehow fail to observe that it forms a nine-note scale, particularly since the pattern is formed out of three groups of three notes and is even beamed to show this structure. (With this pattern it is quite obvious that pitch-classes are not repeated, partly because they appear in clusters.)

A curious dynamic results. If Coltrane consciously used pattern 286 in the second half, then his use of the nonatonic collection in the first half is more likely to have been coincidental, as there are fewer notes to account for. However, his use of the nonatonic collection in the first half is at the same time less likely to have been coincidental, due to his specific choice of pattern 286. Similarly, in the first half of GS, Coltrane's use of the same third-related key areas and his sequencing by major third reduces the number of notes to account for, but also suggests awareness of a nonatonic collection (since any pattern in a nonatonic collection may be repeated up or down a major third and will stay within the same collection). It is possible, but unlikely, that Coltrane consciously used pattern 286 and incorporated these elements of design yet was not aware of using a nonatonic collection. If Coltrane did not consciously use pattern 286, then he almost certainly would have been aware of his consistent use of the same nine-note collection.

Such awareness of compositional structure is often attributed to twentieth-century composers in the Western European tradition, perhaps in part due to the formal training they typically received. However, Coltrane studied theory intensively with Dennis Sandole at the Granoff School in Philadelphia for several years.[30] In addition, there are documents that point to his deep interest in theory,[31] and there are other Coltrane works exhibiting a level of organization that requires awareness of theoretical principles.[32]

[30]According to Lewis Porter, "Coltrane was at Granoff for over four years (interrupted by stints on the road)." Porter, *John Coltrane: His Life and Music* (Ann Arbor: University of Michigan Press, 1998), 52. For more on the material Sandole covered with Coltrane, see Demsey, "Chromatic Third Relations," 153–4; and Porter, *John Coltrane*, 51, 146–7.

[31]See, for example, the drawings appearing in the front matter of Yusef Lateef, *Repository of Scales and Melodic Patterns* (New York: Alfred, [1981] 2015). Originally published by Fana Music (Amherst, MA). Coltrane's drawings may have stemmed from his work with Sandole, as Pat Martino also studied briefly with Sandole and drew similar diagrams. See Guy Capuzzo, "Pat Martino's *The Nature of the Guitar*: An Intersection of Jazz Theory and Neo-Riemannian Theory," *Music Theory Online* 12, no. 1 (2006).

[32]For example, John O'Gallagher's article appearing later in this issue demonstrates that Coltrane and his wife Alice Coltrane (on piano) adhere strictly to the same trichordal set class throughout their performance of "Iris"; other significant cases of set-class organization in Coltrane's late period are presented as well. John O'Gallagher, "Pitch-Class Set Usage and Development in Late-Period Improvisations of John Coltrane," In "Coltrane at Fifty," special issue, *Jazz Perspectives* 12, no. 1 (2020): 93–121.

Part II: Voice Leading

Zero-Sum Voice Leading in "Giant Steps"

Santa's nonatonic systems are an extension of Cohn's hexatonic systems.[33] Each of Santa's systems contain three major triads and three dominant-seventh chords – with the fifths of the seventh chords omitted to preserve cardinality – cycling through three key areas related by major thirds. Movement between any two chords within these nonatonic systems yields a parsimonious voice-leading sum (PVLS) of zero and a directed voice-leading sum (DVLS) of zero.[34]

Santa's PVLS is a variation of Cohn's DVLS.[35] Voice leading is idealized (abstract) when using either.[36] PVLS uses directed pitch intervals to measure the most efficient voice leading between two chords, then takes the absolute value of the result (i.e. converting any negative results to positive). PVLS indicates how much total motion in one direction is required to move between two chords, but does not specify the direction of the motion. (Component motions are directed, "canceling out" contrary motion, but the overall motion is undirected.)[37]

In this article I am usually concerned with the direction of the overall motion required to move between two chords, and will represent ideas using clockfaces as well. For both of these reasons, I will use Cohn's DVLS rather than Santa's PVLS. These directed voice-leading sums also represent the total amount of voice-leading motion required to move from one chord to another in one direction, with component motions in opposite directions – clockwise or counter-clockwise in this case – canceling each other out. Since chords are considered abstractly, in pitch-class space, this overall directed motion is indicated via a directed pitch-class interval.[38]

One way to calculate DVLS values is to pair each note of the first chord to a note of the second chord, then add the directed pitch-class intervals involved in moving from the first chord to the second, modulo 12. An example of this procedure will be seen in Figure 11. (The DVLS will be identical, regardless of how the voices are paired.) Another way to calculate the DVLS is to sum the pitch-classes of each chord, then subtract the sum of the first chord from the sum of

[33] Cohn, "Maximally Smooth Cycles."

[34] Zero-sum voice leading results when the amount of voice-leading motion in one direction is balanced by the same amount of voice-leading motion in the opposite direction. This can be seen in Figure 11. More detailed explanations follow.

[35] Santa, "Nonatonic Progressions," 15–16; Cohn, "Square Dances with Cubes," in "Neo-Riemannian Theory," special issue, *Journal of Music Theory* 42, no. 2 (1998), 285–7.

[36] For more on idealized voice leading, see Cohn, *Audacious Euphony*, 6.

[37] Santa's PVLS is distinct from Cohn's "voice-leading efficiency" (VLE), Joseph Straus's "total displacement," and Steven Rings's absolute voice-leading sums (AVLS). Cohn, "Square Dances," 283–4; Joseph N. Straus, "Uniformity, Balance, and Smoothness in Atonal Voice Leading," *Music Theory Spectrum* 25, no. 2 (2003): 321–2; Rings, "Riemannian Analytical Values, Paleo- and Neo-," in *The Oxford Handbook of Neo-Riemannian Music Theories*, ed. Edward Gollin and Alexander Rehding (New York: Oxford University Press, 2011), 490.

[38] Cohn's full formalization of DVLS is given for triads, but his extension of sum-class transformations to seventh chords suggests that the same principles of summation should apply for any two chords of the same cardinality. Nevertheless, I calculated the DVLS values in this article in multiple ways to be sure that there were no exceptions among the chord relationships. Cohn, "Maximally Smooth Cycles," 294–5.

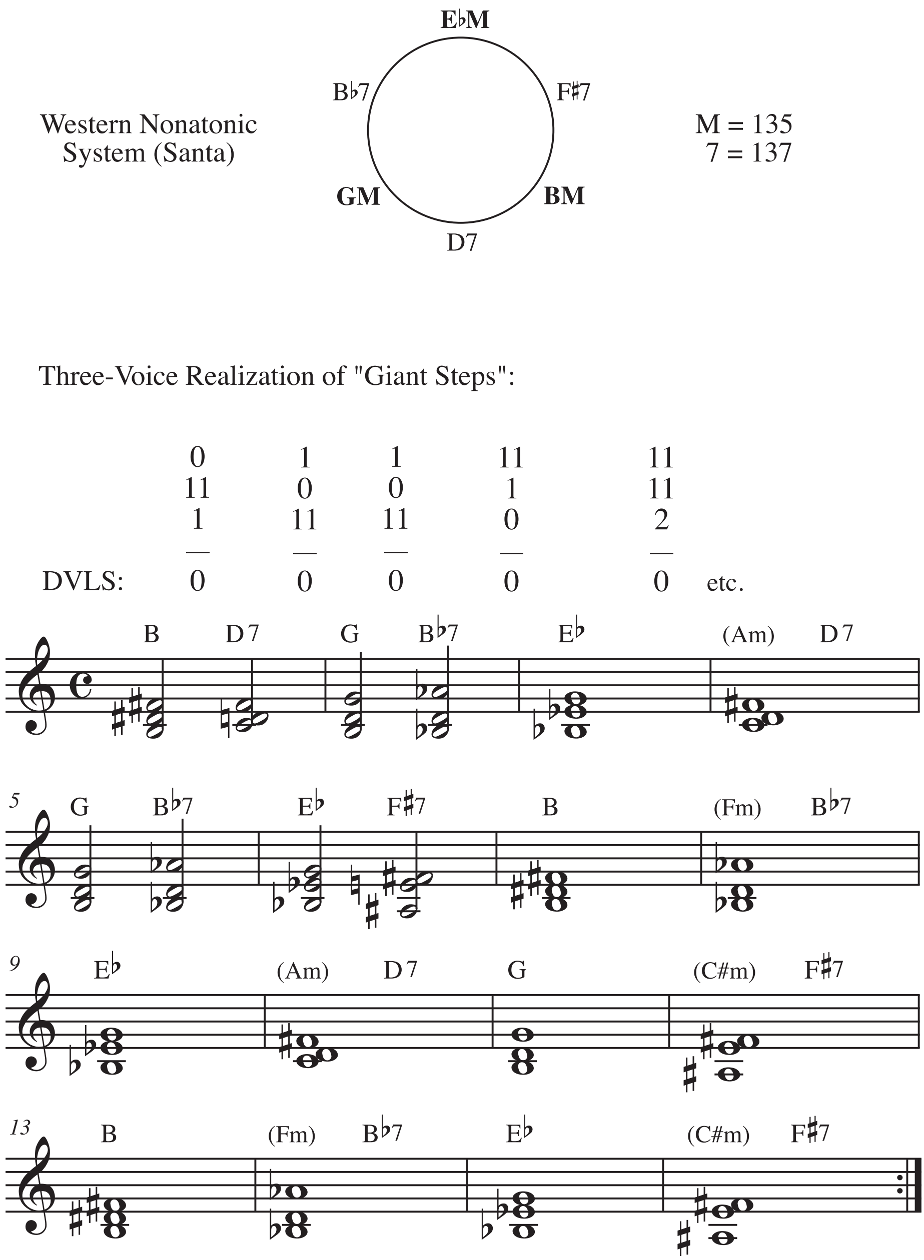

Figure 11. Santa's Western nonatonic system (above), and a three voice, zero-sum realization of "Giant Steps" using this system (below). Constituent voice-leading motions are indicated above the staff and sum to zero, modulo 12. Santa, "Nonatonic Progressions," 16.

the second chord, modulo 12, in accordance with Cohn's Theorem (1a): DVLS (X,Y) = SUM(Y) – SUM(X).[39] For example, the DVLS from a C major triad to a

[39]Ibid., 286.

C♯ major triad may be calculated thus:

$$(1 + 5 + 8)–(0 + 4 + 7) = 3$$

Santa observes that the harmonies of GS correspond to those in the Western nonatonic system. However, he is unable to account for all of them since there are no minor chords in the nonatonic systems. He does not address this issue, but all of the minor chords in GS are part of ii-V-I progressions, and may therefore be regarded as elaborations of the dominant chords, or even as interchangeable with them. The interchangeability of the ii and V chords has been widely discussed in pedagogical literature, particularly in sections dealing with chord substitution or the use of bebop scales.[40] This concept is especially applicable to GS due to its rapid tempo and to the harmonic motion itself, which provides an abundance of chromatic interest that lessens the need to alter the dominant chords.

Many of the chord progressions in GS move in order around the Western system (pictured in Figure 11), but chords in these systems do not have to be adjacent in order to have an underlying (idealized) voice-leading sum of zero. Therefore, all of the chords in GS can be connected with zero-sum voice leading, creating a seamless quality.

Figure 11 illustrates this with a three-voice realization of GS that removes the minor chords, thus using the Western system exclusively. Chord voicings employed in the realization – 135 (root, third, fifth) for major chords and 137 for dominant chords – are indicated to the right of the system. (Voicings are similarly indicated in subsequent diagrams.) Constituent voice-leading motions are indicated above the staff using directed pitch-class intervals and produce a DVLS of zero in all cases. For example, as the first chord moves to D7, the upper voice is retained, while the lower voices move in opposite directions by half step, the voice-leading motions of 0, 1, and 11 adding to zero.

Figure 12 takes the voice-leading relationships of Figure 11 and expresses them with a voice-leading zones diagram following Cohn.[41] Voice-leading zones can be helpful in visualizing the zero-sum concepts of this article. The numbers, presented as a clock face, are underlined to indicate that zones are represented, as opposed to pitch-classes. The pitch-classes of any chord in a particular zone sum to the number of the zone itself. Therefore, all chords in the same zone move to one another using zero-sum voice leading.

Figure 12 illustrates the fact that Santa's nonatonic systems each remain within one voice-leading zone. The Western nonatonic system falls within zone 8. Correspondingly, we can see that the pitch classes comprising each of

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[40]For example, see David Baker, *How to Play Bebop, Volume 1: The Bebop Scales and Other Scales in Common Use* (Van Nuys, CA: Alfred, 2006, originally published in Bloomington, IN: Fragipani Press, 1985), 1; and Mark Levine, *The Jazz Theory* Book (Petaluma, CA: Sher Music, 1995), 173–4.

[41]Cohn, *Audacious Euphony*, 104. Douthett and Steinbach's "Cube Dance," which appears superimposed over the clock face in Cohn's diagram, is itself not necessary for this discussion. However, readers familiar with Cohn's diagram, including "Cube Dance," may find further discussion of it in note 7.

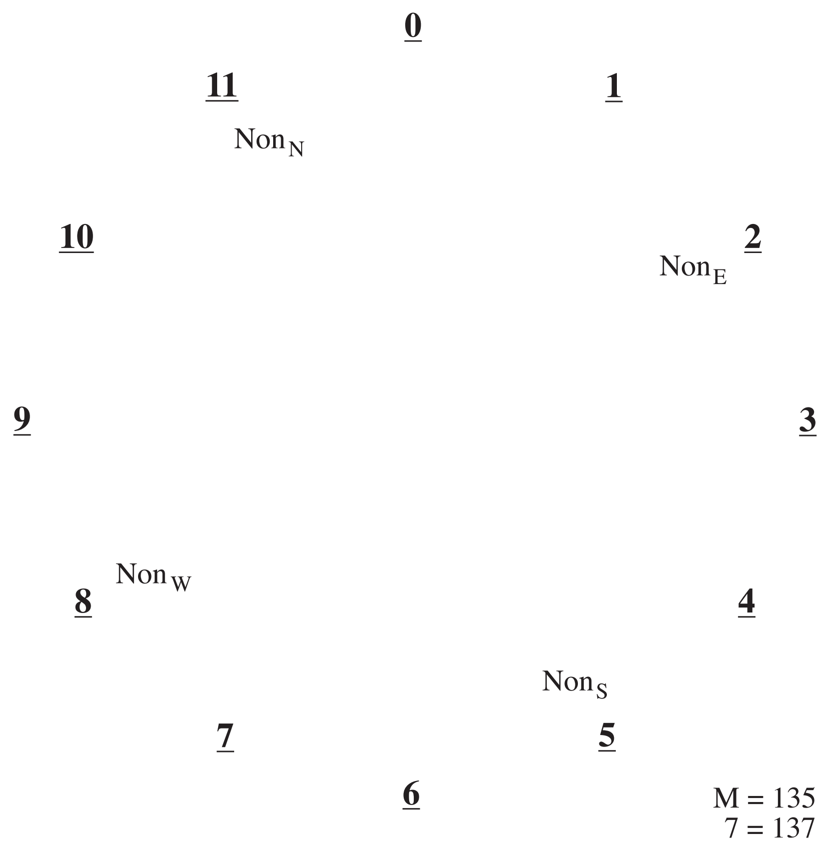


Figure 12. Voice-leading relationships among Santa's nonatonic systems, expressed with a voice-leading zones diagram following Cohn.

the first four chords from the Western nonatonic realization in Figure 11 sum to 8, modulo 12:

$$(11 + 3 + 6) = (0 + 2 + 6) = (11 + 2 + 7) = (10 + 2 + 8) = 8_{\text{mod}12}$$

The same is true for any of the chords in Figure 11.

Figure 12 also shows the voice-leading relationships between different nonatonic systems. While chords in the same zone lead to one another with a DVLS of 0, chords in adjacent zones lead to one another with a DVLS of 1 or 11. Chords in different nonatonic systems will lead to one another with a DVLS of 3, 6, or 9. For example, chords in zone 5 (where Non_S resides) may move to chords in zone 8 (where Non_W resides) with a DVLS of 3.

One might object to Figures 11 and 12 on grounds that the ii chords are not accounted for (the above remarks notwithstanding), nor are the major sevenths of the I chords accounted for. Ideally, an abstract realization of GS in this vein would include ii chords, include the major sevenths, feature zero-sum voice leading throughout, and use three or more voices. I address these issues in the discussion that follows, where I present a number of diagrams that are similar to Figure 11, using different voicings or collections as the basis for the GS realization. I have found that the abovementioned criteria cannot be fulfilled all at once, but that various combinations of them may be fulfilled.

Before proceeding I would like to make four comments on the diagrams that follow. First, none of the diagrams feature zero-sum voice leading between every chord, and therefore the voicings often follow the major-third cycles around the staff, keeping the voice leading consistent for the sake of clarity. (In practice, a musician would more likely change the voicing used for the different iterations of V-I and ii-V-I progressions, allowing the chords to stay in the optimal register.) Second, the diagrams create new hexatonic and nonatonic systems by changing the voicings of the chords or by changing the roots of the chords. This alters the properties of the circular arrangements by Cohn and Santa. (The properties of their systems also differ from each other). Third, most of the diagrams use the left-hand shell voicings popular during the bop era. These voicings consist of the root and either the third or seventh of the chord, with a third tone sometimes added.[42] Lastly, a broader objection to these diagrams might be that the chords are voiced in a way that does not reflect jazz practice (except for the left-hand shell voicings), where chordal extensions and alterations are typical.

Elsewhere I have addressed this broader question – first raised by Steven Strunk – with respect to neo-Riemannian analysis of jazz in general.[43] After developing an approach that models actual voicings separately from the underlying harmonies they express, I conclude that the distinction between these two layers is more significant for the modal/post-bop repertoire discussed by Strunk, Keith Waters, J. Kent Williams, and Joon Park, and presents no issue for the analysis of GS, which is functionally tonal (despite its three-key scheme).[44] At the same time, I do consider the major-seventh chord in the present essay.[45]

[42]The third is often played as a tenth. In the tonal context of bebop, these shell voicings imply the harmony sufficiently and free up the powerful tenor range of the piano for right hand solo ideas (as opposed to filling that range with a stride chord, or, in later practices, a rootless left-hand voicing).

[43]Rich Pellegrin, "Modeling Salience and Stability: A New Perspective on Neo-Riemannian Theory" (paper delivered at the City Music Analysis Conference, London, England, July 5–8, 2018); Steven Strunk, "Wayne Shorter's 'Yes and No': An Analysis," *Tijdschrift voor Muziektheorie* 8, no. 1 (2003): 40–56. Prior to Strunk's article, Clifton Callender discussed transformations between extended chordal structures. Clifton Callender, "Voice-Leading Parsimony in the Music of Alexander Scriabin," in "Neo-Riemannian Theory," special issue, *Journal of Music Theory* 42, no. 2 (1998): 219–33; Clifton Callender, "Interactions of the Lamento Motif and Jazz Harmonies in György Ligeti's *Arc-en-ciel*," *Intégral* 21 (2007): 41–77.

[44]Strunk, "Shorter's 'Yes and No'"; Steven Strunk, "Notes on Harmony in Wayne Shorter's Compositions, 1964–67." *Journal of Music Theory* 49, no. 2 (2005): 301–32; Steven Strunk, "Tonal and Transformational Approaches to Chick Corea's Compositions of the 1960s," *Music Theory Spectrum* 38, no. 1 (2016): 16–36; Keith Waters, "Chick Corea and Postbop Harmony," *Music Theory Spectrum* 38, no. 1 (2016): 37–57; Keith Waters and J. Kent Williams, "Modeling Diatonic, Acoustic, Hexatonic, and Octatonic Harmonies and Progressions in Two- and Three-Dimensional Pitch Spaces; or Jazz Harmony after 1960," *Music Theory Online* 16, no. 3 (2010); Joon Park, "Reflections on (and in) Strunk's *Tonnetz*," *Journal of Jazz Studies* 11, no. 1 (2016), special issue in honor of Steven Strunk: 40–64.

[45]Much neo-Riemannian work addressing seventh chords has omitted discussion of the major-seventh chord. This includes several articles from the 1998 special issue of the *Journal of Music Theory* (42, no. 2) devoted to neo-Riemannian theory – Jack Douthett and Peter Steinbach, "Parsimonious Graphs"; Edward Gollin, "Some Aspects of Three-Dimensional '*Tonnetze*'"; and Adrian Childs, "Moving Beyond Neo-Riemannian Triads: Exploring a Transformational Model for Seventh Chords" – related work such as Richard Bass, "Half-Diminished Functions and Transformations in Late Romantic Music," *Music Theory Spectrum* 23, no. 1 (2001): 41–60; and Bass, "Enharmonic Position Finding and the Resolution of Seventh Chords in Chromatic Music," *Music Theory Spectrum* 29,

For the purposes of this article, it should simply be observed that the diagrams provide abstract representations of the underlying (idealized) voice leading of GS, and do not represent how chords are actually realized in jazz practice.

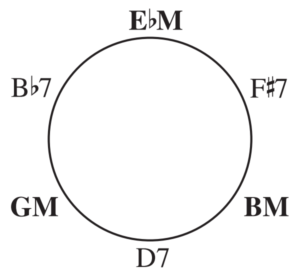
Incorporating the major sevenths of the tonic chords while staying within the Western system may be accomplished by substituting the major seventh for the third, redefining Santa's original system (see Figure 13). The new chord voicings are shown to the right of the system. Some of the zero-sum properties of Santa's system are retained with this redefinition, as shown in the diagram. However, the ii chords continue to be omitted and the voice leading is suboptimal – the sevenths in the dominant chords are unresolved, and it would be rare for a jazz player to voice chords in this manner. On the other hand, there is in fact nowhere for the sevenths to resolve in this configuration, and again, modeling practice is not the function of this inquiry (it would also be rare to voice chords as in Figure 11).

The new voicing for the major chord (157) retains the property of zero-sum voice leading with other major chords in the system. The dominant chord voicings – which are unchanged in this configuration – also relate to one another with zero-sum voice leading, but sum to a different value than the major chords. Since every motion in GS (excluding the minor ii chords) alternates between major and dominant, the resulting series of DVLS values is 5, 7, 5, 7, etc. Figure 14 expresses these voice-leading relationships with a voice-leading zones diagram.

Figure 15 shows a realization of GS that incorporates the ii chords, and still stays within a single nonatonic collection. However, the number of voices has been dropped from three to two. As can be seen at (b), we are now working with the Southern nonatonic collection, which is equivalent to the set of all melody notes and all chord roots of GS. Santa's Southern nonatonic system, at (a), contains the tonal centers of D major, F♯ major, and B♭ major. At (c), I have rearranged the pitch classes of the Southern nonatonic collection to create a new system containing the tonal centers of E♭ major, G major, and B major, and the ii chords have been added in. Had I left the ii chords out, this Southern system would be identical in layout to Santa's Western system; this shift is made possible due to the fact that I have also redefined the chord voicings (as shown to the right of the new system).

no. 1 (2007): 73–100; and contributions involving cross-type transformations such as Julian Hook, "Uniform Triadic Transformations," *Journal of Music Theory* 46, no. 1–2 (2002): 57–126; Julian Hook, "Cross-Type Transformations and the Path Consistency Condition," *Music Theory Spectrum* 29, no. 1 (2007): 1–39, and Guy Capuzzo, "Neo-Riemannian Theory and the Analysis of Pop-Rock Music," *Music Theory Spectrum* 26 no. 2 (2004): 177–99. Major-seventh chords are addressed in Dmitri Tymoczko, *Geometry of Music*; Samuel Reenan and Bass, "Types and Applications of $P_{3,0}$ Seventh-Chord Transformations in Late Nineteenth-Century Music," *Music Theory Online* 22, no. 2 (2016); and Callender, "Ligeti's *Arc-en-ciel*." (Tymoczko's *Geometry of Music* is not explicitly a neo-Riemannian work, and is also heavily influenced by jazz theory.) Major-seventh chords are also mentioned briefly in Jack Douthett, "Filtered Point-Symmetry and Dynamical Voice-Leading," in *Music Theory and Mathematics: Chords, Collections, and Transformations*, eds. Jack Douthett, Martha M. Hyde, and Charles J. Smith (Rochester, NY: University of Rochester Press, 2008), 79. Several jazz-focused articles address the major-seventh chord from a neo-Riemannian perspective, including Michael McClimon, "Transformations in Tonal Jazz: ii-V Space," *Music Theory Online* 23, no. 1 (2017); and those mentioned above.

Western Nonatonic System, Redefined



M = 157
7 = 137

Voice-leading properties of the redefined system

Motion	DVLS
any M7 to any M7	0
any Dom7 to any Dom7	0
any M7 to any Dom7	5
any Dom7 to any M7	7

Three-Voice Realization of "Giant Steps":

Figure 13. Santa's Western nonatonic system redefined with different chord voicings (above), the voice-leading properties of the redefined system (middle), and a three-voice realization of "Giant Steps" using this system (below).

In addition to dropping to two voices here, the zero-sum voice-leading property of Santa's system is sacrificed. This realization therefore focuses on what Demsey refers to as aspects of "pitch control" in his and Jaffe's original

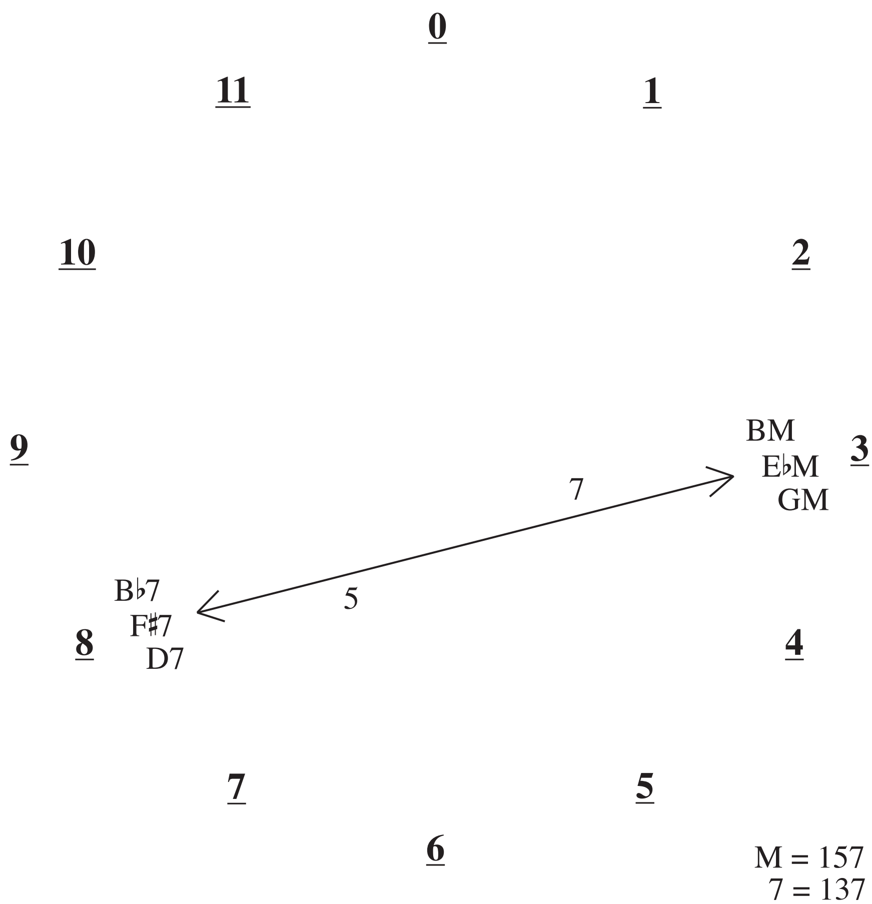


Figure 14. The voice-leading relationships from Figure 13 expressed with voice-leading zones.

observations – the limitation to a certain pitch-class collection, in this case a symmetrical scale that contains all the melody notes and chords roots.[46] An additional feature of the Southern system is that it accounts for the (1,3) whole-tone collection used by bassist Paul Chambers during the first half of the head of GS on the original recording (see Figure 16). This whole-tone aspect is yet another piece of the GS puzzle.

A Hexatonic Realization

Common to both the Western and Southern nonatonic collections is the Western hexatonic collection. This hexatonic core suggests that some sort of hexatonic realization should be possible. Figure 17 shows Cohn's Western hexatonic system, a rearrangement of the pitch classes of the Western hexatonic collection to create a new system containing the chord changes of GS, and a

[46]Demsey, "Chromatic Third Relations," 172.

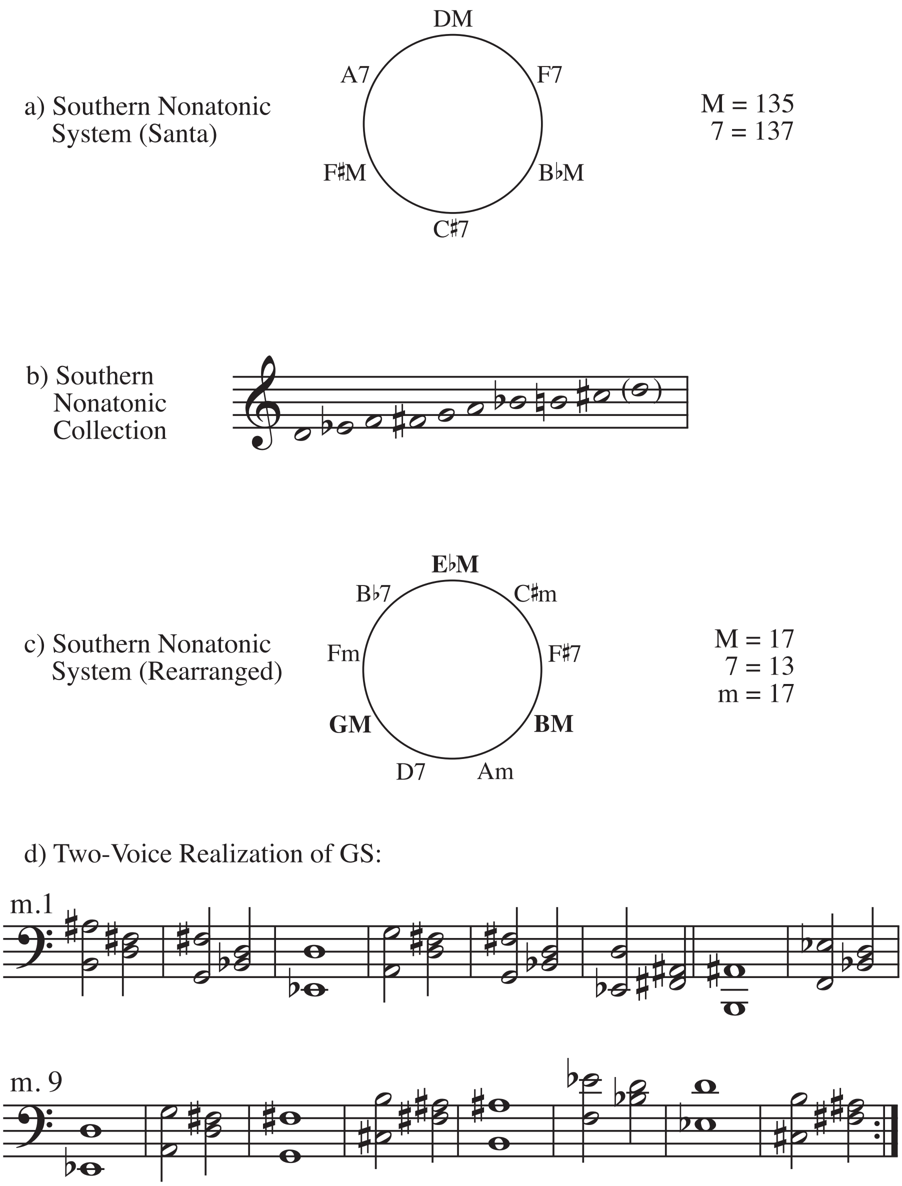


Figure 15. Santa's Southern nonatonic system (a), the Southern nonatonic collection (b), a rearrangement of the pitch classes of the Southern nonatonic collection to create a new nonatonic system (c), and a two-voice realization of "Giant Steps" using this system (d).

two-voice realization of GS using the new system.[47] This rearranged hexatonic system itself is visually identical to Santa's Western nonatonic collection, but I have also redefined the chord voicings here to create a hexatonic system. This realization again uses two-note shell voicings and also omits the ii chords, since the roots of the ii chords do not belong to $Hex_{(2,3)}$.

[47]Cohn, "Maximally Smooth Cycles."

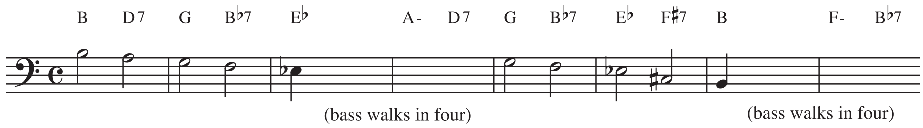


Figure 16. The (1,3) whole-tone bassline performed by Paul Chambers during first half of the head of "Giant Steps."

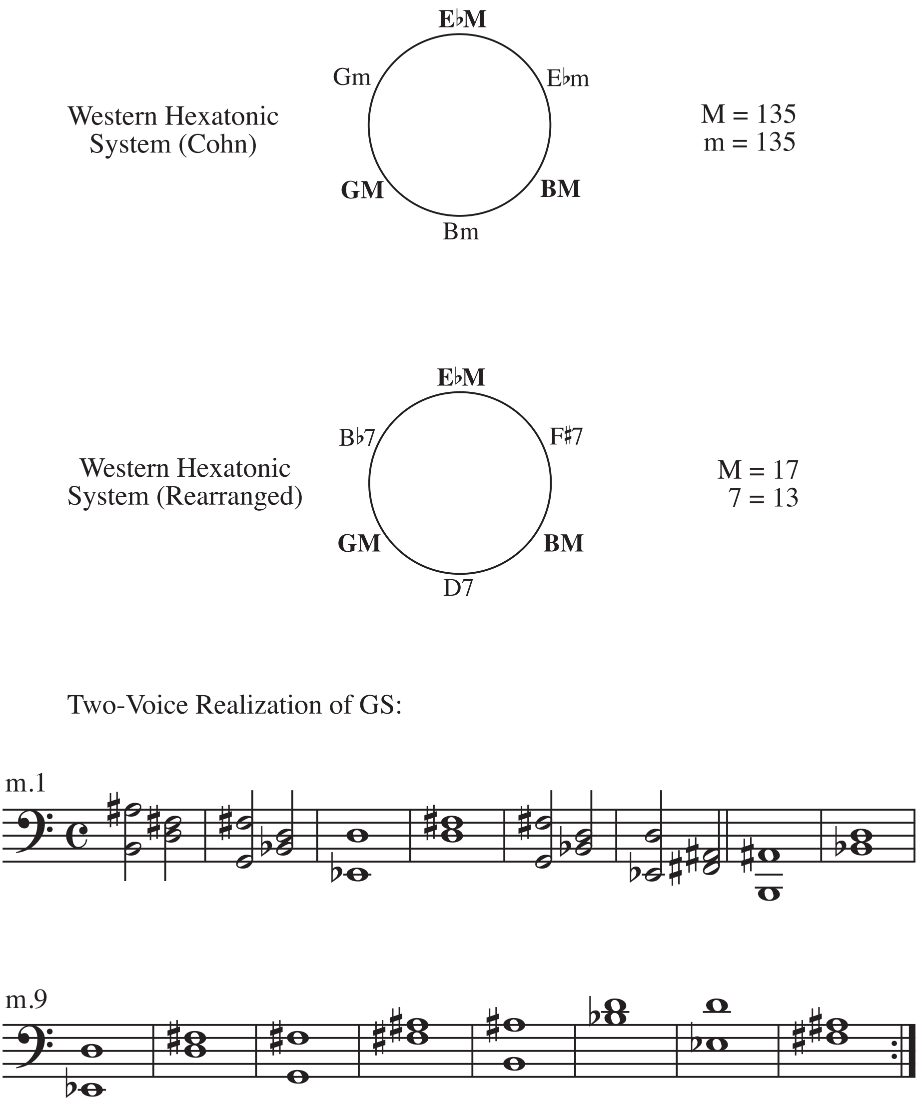


Figure 17. Cohn's Western hexatonic system (above), a rearrangement of the pitch classes of the Western hexatonic collection to fit the chord changes of "Giant Steps" (middle), and a two-voice realization of "Giant Steps" using the new system (below). The Western hexatonic collection is common to both the Western and Southern nonatonic collections.

The hexatonic core of GS has clear surface-level significance as well. The first half of the melody is composed exclusively of this collection, save for one occurrence of A at the end of measure four. This A is the second pitch of the (025) trichordal cell used throughout the second half of the tune.[48] The second pitches of these cells – A, C♯, and F – are the only melodic tones in the composition that do not belong to $\text{Hex}_{(2,3)}$, and Coltrane sometimes replaces these notes with common tones that do belong to $\text{Hex}_{(2,3)}$.[49] These three pitches are also the roots of the ii chords (adding them to $\text{Hex}_{(2,3)}$ yields the Southern nonatonic collection).[50] Only two roots in the first half of GS lie outside of $\text{Hex}_{(2,3)}$, as there are only two ii chords in the first half.[51] The first half of GS is thus almost entirely hexatonic.

It is worth observing the improvisational implications of the hexatonic sound of the first half of GS. One may improvise on the first half of the tune using $\text{Hex}_{(2,3)}$, with the few resulting "outside" tones creating rich dissonances that reflect deeper aspects of composition.[52] Figure 18 gives a sample improvisation, using two-note shell voicings in the left-hand and $\text{Hex}_{(2,3)}$ in the right hand.

Cross-Type Transformations

Considering ic4 cycles vis-à-vis hexatonic collections leads to other significant new terrain. Given that a hexatonic collection contains three complete major-seventh chords, and the fact that the hexatonic core of GS belongs to both the Western and Southern nonatonic systems, it would seem possible to create a more satisfying realization of GS that includes the major sevenths. Complete dominant-seventh chords with the appropriate roots do not exist in either nonatonic system (they are found in the Eastern system), but cross-type transformations of the cardinality variety (following Hook) may be fruitfully explored.[53] Table 1 displays the results of such inquiry.

The center portion of the table works through all of the tonic-dominant motions available in a nonatonic system, with various possible doublings of the dominant chord. Specific major to dominant (M–D) and dominant to major (D–M) motions are indicated with the directed pitch-class interval from the root of the first chord to the root of the second chord (e.g. M–D, 11). The DVLS values resulting from

[48]Here I consider the first half of GS to be the portion containing the prolongation of B (following Demsey, Martin, and Waters). The modified Slonimsky motive (025) from the second half of GS occurs once in the middle of the first half, and also as part of a phrasal overlap with the second half (more below on this).

[49]The C♯, for example, only occurs on one of four iterations of the head on the originally released take – the repeat of the outhead. During the inhead and the first time through the form during the outhead, Coltrane plays a common tone instead, D♯.

[50]Adding C, E, and A♭ to the $\text{Hex}_{(2,3)}$ yields the Western nonatonic collection, and these three tones provide the sevenths of the dominant chords.

[51]The number of ii chords is the same whether one considers the first half to be mm. 1–8 or mm. 16–7.

[52]This approach might work especially well for horn players performing in piano-less trios, an increasingly common context.

[53]Hook, "Uniform Triadic Transformations"; Hook, "Cross-Type Transformations." The relevant, complete ii chords are found in the Northern system. "Cardinality" here refers to the number of distinct elements in a chord *type*, whereas Santa uses "cardinality" to refer to the number of distinct elements in a chord *voicing*. I use the latter sense of the word in note 25.

Figure 18. Sample hexatonic improvisation using hexatonic shell voicings. (The bass note A lies outside of the Hex(2,3) collection).

these root motions – when paired with each of the dominant voicings a–f – are provided beneath. The 135 dominant voicings (voicings a–c) derive from the Southern nonatonic collection, whereas the 137 dominant voicings (voicings d–f) derive from the Western nonatonic collection. A 157 dominant voicing is not available, as it is contained by neither collection. (Again, both the Western and Southern collections contain the 1357 major-seventh chord voicing.)

Using the upper-left cell of the table as an example, a DVLS of 7 results when moving from a dominant-seventh chord with voicing a – 135, doubling the root (1135) – clockwise five semitones to a major-seventh chord (D–M, 5). If the dominant-seventh chord were C7, then the major-seventh chord would be FM7, and the DVLS may be calculated using the pitch-class summing procedure enumerated above:

$$(5 + 9 + 0 + 4)–(0 + 0 + 4 + 7) = 6–11 = 7_{\text{mod12}}$$

Table 1. Cross-type transformations in nonatonic systems.

		Dominant to major			Major to dominant			ic4a cycle	ic4b cycle
								M–D, 3	M–D, 11
	Dom voicing	D–M, 5	D–M, 9	D–M, 1	M–D, 7	M–D, 3	M–D, 11	D–M, 5	D–M, 5
Southern	(a) 135, dbl 1	7	11	3	5	1	9	1, 7 …	9, 7 …
	(b) 135, dbl 3	3	7	11	9	5	1	5, 3 …	1, 3 …
	(c) 135, dbl 5	0	4	8	0	8	4	8, 0 …	4, 0 …
Western	(d) 137, dbl 1	4	8	0	8	4	0	4, 4 …	0, 4 …
	(e) 137, dbl 3	0	4	8	0	8	4	8, 0 …	4, 0 …
	(f) 137, dbl 7	6	10	2	6	2	10	2, 6 …	10, 6 …

D = Dominant seventh 135 or 137 with various doublings. M = Major seventh 1357 voicing.

The right side of the table combines selected root motions, creating two tonic-dominant ic4 cycles that I have labeled ic4a and ic4b.[54] It may at first blush seem that only one ic4 cycle is involved in GS; after all, in the first half of the tune we work our way downwards through major-third cycles, and in the second half we work our way up. However, this does not translate to simple clockwise and counterclockwise motion through nonatonic systems, even though the ii chords have been removed in most of them. Both halves of GS contain V-I motions, but the approach to the V chords differs. In the first half, V chords are approached from a major chord three semitones away (ic4a cycle), whereas in the second half, V chords are approached from a major chord one semitone away (ic4b cycle).[55] The two ic4 cycles of GS may thus be defined as follows (M = major-seventh chord, D = dominant-seventh chord):

$$\text{ic4a cycle} = (\text{M–D}, 3), (\text{D–M}, 5), \ldots$$

$$\text{ic4b cycle} = (\text{M–D}, 11), (\text{D–M}, 5), \ldots$$

The entries for the ic4a and ic4b cycles (in the right-hand columns) together always add to zero, since they are inverses of one another, the ic4a cycle leading us downwards through major thirds and the ic4b cycle leading us upwards through major thirds. For example, with dominant voicing a, the ic4a cycle proceeds (1,7 …) whereas the ic4b cycle proceeds (9,7 …). $(1 + 7) + (9 + 7) = 24 = 0_{\text{mod } 12}$. This suggests that GS steps has a sort of global zero-sum logic to it, but the same is true of any repeated form, as what goes up must come down. Rather, we are interested in finding zero-sum patterns of a more localized nature.

With respect to the ic4a cycles of the first half of GS, none of the possible dominant voicings yield true, or truly local, zero-sum voice leading; i.e. no cells in this column read "0,0." Voicing d appears to produce the most interesting results, as the DVLS will always be 4, creating a voice-leading cycle that occurs on a more local level than the ic4 key cycle.[56] Figure 19 shows a realization of an ic4a cycle using another redefinition of the Western nonatonic system, this time employing cross-type chord voicings, including dominant voicing d. The cycle starts on B to recall GS. The upper voices each progress canonically through a series of directed pitch intervals – 0, 0, 0, +2, −1, −1 – and through a pathway of chord tones: root, fifth, third, seventh, seventh, third.[57] (The bass simply leaps from root to root.) The canonic entrances are

[54]Masaya Yamaguchi identifies four types of tonic-dominant ic4 cycles, but it is unclear what the difference between M.T.C. 2a and 2b are, as both have subtypes ascending and descending through key areas by major thirds. Masaya Yamaguchi, "A Creative Approach to Multi-Tonic Changes: Beyond Coltrane's Harmonic Formula," *Annual Review of Jazz Studies* 12 (2002): 157.

[55]This half-step root motion requires a skip to the opposite nonatonic pole. As Santa observes, two chords in a polar relationship have no common tones, as is also the case with Cohn's hexatonic poles. Santa, "Nonatonic Systems," 7; Cohn, "Maximally Smooth Cycles."

[56]The consistent paths exhibited here are not the same as Hook's "path consistency." Hook, "Cross-Type Transformations."

[57]There is no particular significance to the canon starting with zero. I chose to begin with the zeros because they are easy to see. The point is simply that the same voice-leading pattern rotates throughout the upper voices.

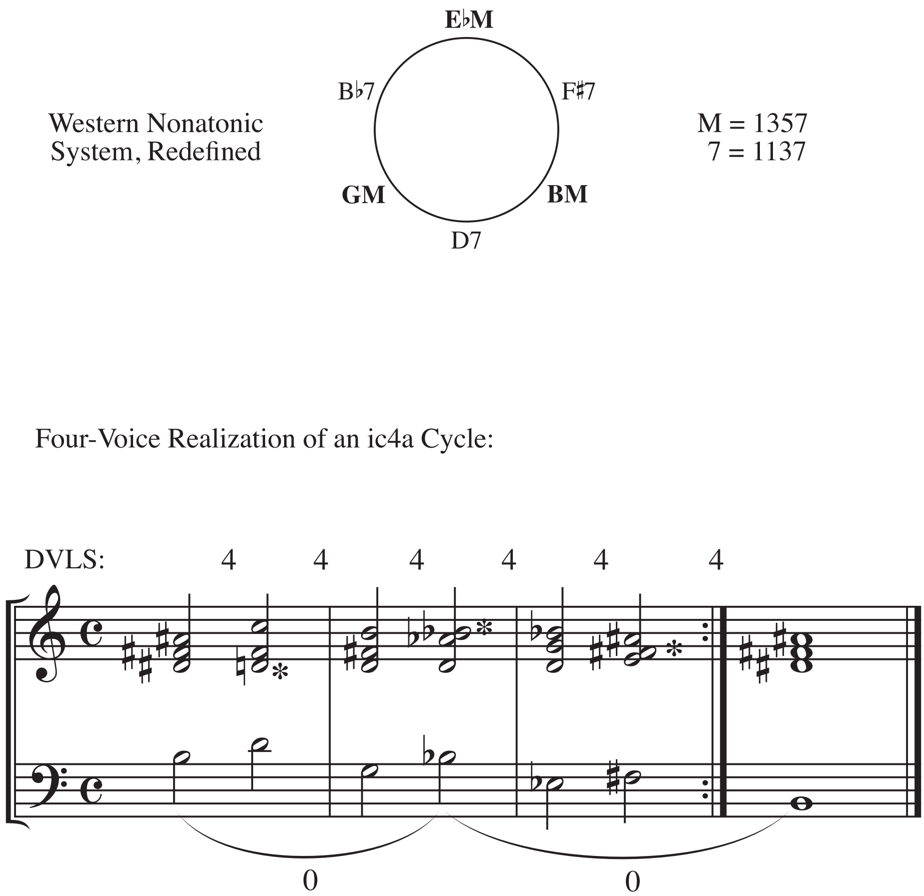


Figure 19. Santa's Western nonatonic system redefined with cross-type voicings (including dominant voicing d) from Table 1 (above), and a four-voice realization of an ic4a cycle using this system (below). Canonic entrances are denoted with asterisks.

denoted with asterisks. For example, the asterisk in the fourth chord appears next to the B♭ in the upper voice, which then proceeds to B♭, A♯, A♯, C, and B before returning to B♭, articulating the patterns described.

More intriguing is the fact that path consistency is only a property of voicing d, which is the voicing typically used. (The 135 voicing may of course be used for a dominant but does not in itself capture the dominant quality the way a 137 voicing does.) The dominant chords also resolve normatively. The only part-writing issue we encounter here is the resolution of the major seventh, which would typically resolve down a half-step to the fifth of the dominant chord, but the fifth is not available in the Western system and thus the major seventh resolves upwards, creating parallel unequal sevenths.

The ic4b column of Table 1 also lacks an entry of (0,0). However, voicing d again suggests an avenue of exploration. The ic4b entry for voicing d is (0,4). Just beneath that, we see the entry for ic4b, voicing e, which is the reverse, (4,0). (Voicing c is also (4,0), but is drawn from a different nonatonic system.)

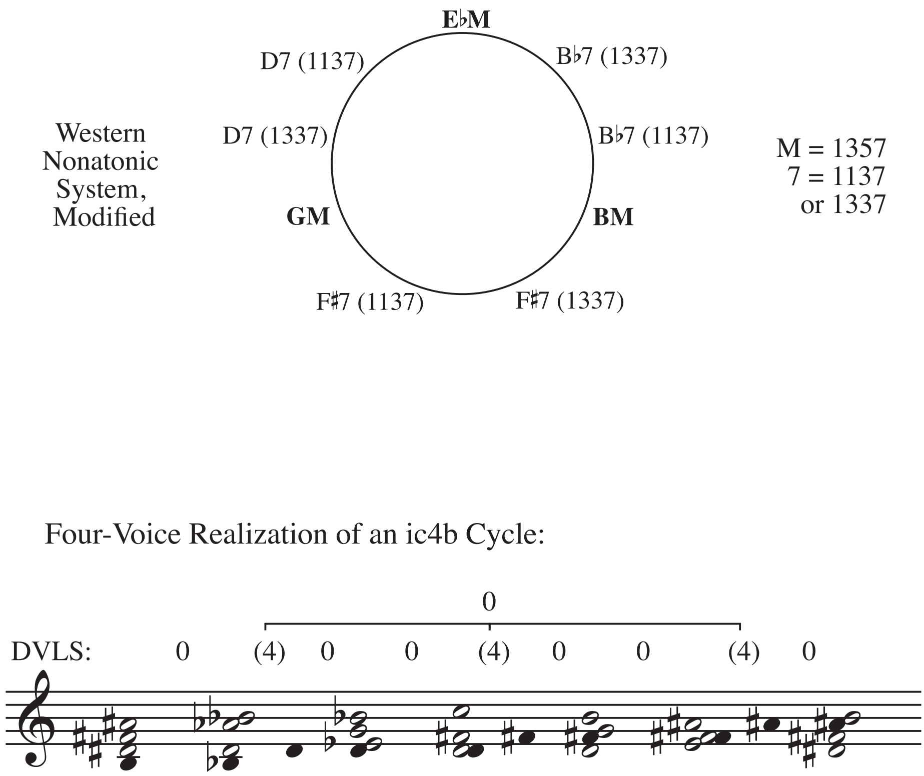


Figure 20. A modification of the redefined system given in Figure 19 to allow for two different dominant voicings (above), and a four-voice realization of an ic4b cycle using this system (below). Accidentals apply to individual notes only.

By using both voicings, a pivot can be created that gets closer to truly local zero-sum voice leading. Figure 20 illustrates this with a modification of the redefined system given in Figure 19 and a four-voice realization of an ic4b cycle that is zero-sum (except for the change of doubling). Because this modified system specifically models the ic4b cycle, I have altered the placement of the dominant chords. While the 1337 voicing may strike us as incorrect, the doubled third reflects the dual function of the seventh scale degree in V-I jazz progressions – that of dominant leading tone which resolves up and that of tonic color tone which is approached by common tone. In addition, the voicings presented here do not represent actual practice but are abstract representations of underlying voice-leading motions (as previously mentioned).

Figure 21 combines Figures 19 and 20 to create a complete four-voice realization of GS (with ii chords omitted) using the Western nonatonic system and cross-type transformations that yield quasi zero-sum voice leading. Figure 22 expresses the voice-leading relationships among all of the chords involved with a voice-leading zones diagram. In Figure 21, GS has been broken into two systems in the diagram layout, divided according to the existing normative hypermetric structure (8 + 8 measures), for simplicity. However, note that the

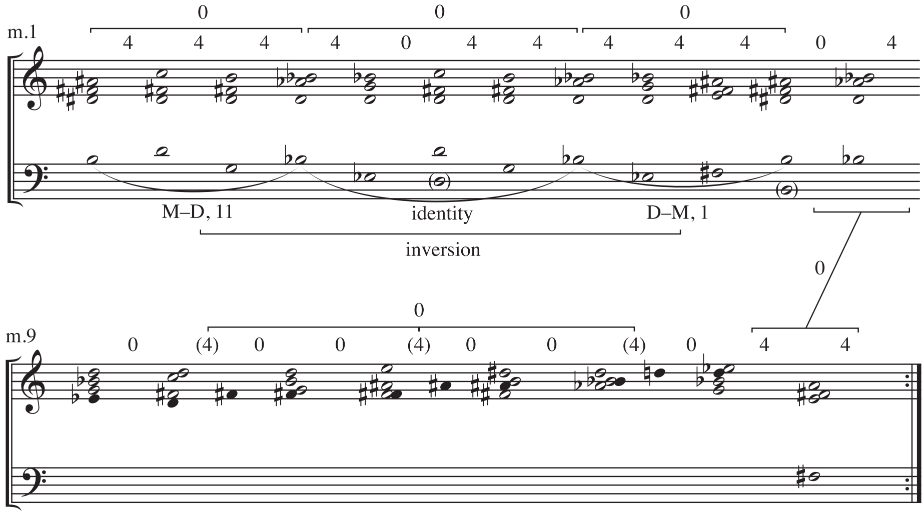


Figure 21. A four-voice realization of "Giant Steps" (ii chords omitted) using the Western nonatonic system and employing cross-type transformations that yield quasi zero-sum voice leading. Accidentals apply to individual notes only. Octave switching in lower voice for consistency with Figures 19 and 20.

prolongational structure first observed by Demsey is one measure out of phase with the hypermetric structure, due to the ii–V progression that prepares each eight-bar hypermetric unit (the latter of these being the turnaround).[58] In addition, the phrase structure between these two eight-bar hypermetric units overlaps by one measure; and, as Demsey observes, this overlapping measure at first seems to be the beginning of another descending major-third transposition of mm. 1–4.[59]

In the first half of GS, which prolongs BM, three segments sum to zero.[60] But while three equidistant key areas are also traversed, the segments and the key areas do not align. The segments are zero-sum as a result of their constituent voice-leading motions of 4, but are also reflected in their beginning and ending chords, as indicated below the first system. These beginning and ending chords are BM7, B♭7, B♭7 and BM7. We know from Table 1 that a root motion of 11 from a major (1357) chord to a dominant (1137) chord yields a DVLS of zero, as does the motion back to the major chord. The zero-sum segments are defined by these motions as BM7 moves down to B♭7, B♭7 moves to itself (identity), and B♭7 then moves back up to BM. The B♭7 chord at the end of the system represents the phraseologically overlapping measure.

The second half of GS (the second system of Figure 21) prolongs E♭, which works out as in Figure 20. The turnaround adds two more voice-leading motions of 4 to the motion of 4 from the end of the first half, summing to zero as the form starts over again. In Figure 21 overall we can see the

[58]Demsey, "Chromatic Third Relations," 170.
[59]Ibid., 169–70.
[60]Prolongation is separate from zero-sum voice leading, but anytime a chord is stated, departed from, and returned to, the overall voice leading will sum to zero.

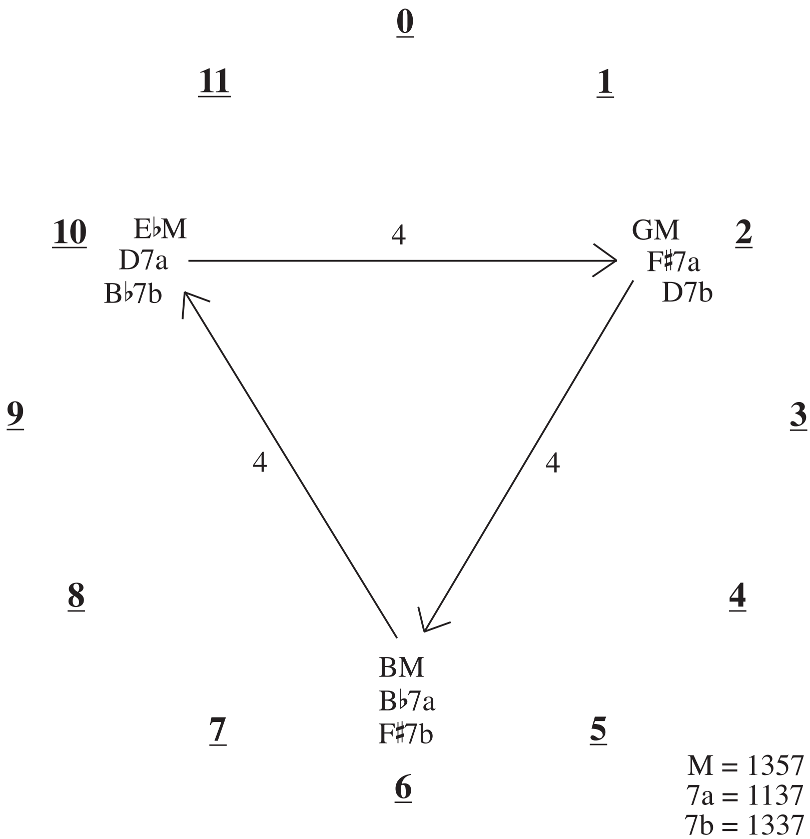


Figure 22. The voice-leading relationships of Figure 20 expressed with voice-leading zones. The three major and three dominant chords of "Giant Steps" are each shown, with the two dominant voicings from Figure 20 represented as "7a" and "7b." Since the chords now contain four voices rather than three, augmented triads are no longer implied in zones 0, 3, 6, and 9.

prolongation of B – BM7 to BM7 – which naturally adds to zero, and the prolongation of E♭ – E♭M7 to E♭M7 – which also adds to zero. The connective parts then themselves add to zero, returning us to where we began.

Table 2 summarizes the GS realizations I have explored above in terms of the four properties we have been concerned with – inclusion of major sevenths, inclusion of ii chords, use of at least three voices, and zero-sum voice leading.[61] "Redefined" systems are those where I have kept the existing chord structure of Santa's or Cohn's systems, but changed the voicings of the chords. "Rearranged" systems are those where I have departed entirely from Santa's or Cohn's systems, creating new systems by rearranging the notes of the associated nonatonic or hexatonic collection (and using new chord voicings as well). The cross-type realization from Figure 21 uses the redefined system from Figure 19 and its modification, given in Figure 20. As can be seen, each

[61]For additional realizations of GS using zero-sum LP cycles as upper structures for chord voicings, see Pellegrin, "Modeling Salience and Stability."

Table 2. Summary of "Giant Steps" realizations. "Redefined" systems are those where the chord voicings have been changed from the original systems. "Rearranged" systems are those created by rearranging the notes of the associated nonatonic or hexatonic collection (and using new chord voicings as well). The cross-type realization from Figure 18 uses the redefined system from Figure 16 and its modification, given in Figure 17.

	System used	Major 7ths	ii chords	3 voices	Zero-sum
Figure 11	Western nonatonic			yes	yes
Figure 13	Western nonatonic, redefined	yes		yes	quasi
Figure 15	Southern nonatonic, rearranged	yes	yes		
Figure 17	Western hexatonic, rearranged	yes			
Figure 21	Western nonatonic, cross-type	yes		yes (4)	quasi

realization – except for the hexatonic version – fully meets two out of the four criteria, in various combinations. In addition, each realization corresponds precisely with a symmetrical nonatonic or hexatonic collection, and sometimes also features symmetrical (zero-sum) voice leading, all of which further evidences the highly organized nature of GS.

Part III: Motive

Melodic and Motivic Aspects of "Giant Steps"

The melody of the first half of GS is significant in its use of two descending major-seventh chord arpeggiations, GM and E♭M, which comprise two out of three of the key centers of GS (these two major-seventh chords alone establish $Hex_{(2,3)}$). Each of these major-seventh chord arpeggiations straddles portions of three different key centers, while at the same time anticipates a full ii-V-I progression in the key of the arpeggiated chord. The opening G major-seventh chord arpeggiation participates in chord changes from B major, G major, and E♭ major, which is then followed by a ii-V-I in G major; the subsequent E♭ major-seventh chord arpeggiation leads into a ii-V-I in E♭, initiating the E♭ prolongation that comprises the second half of the composition.[62]

The two descending major-seventh chord arpeggiations in the melody are also closely related to the local root motions, which outline the same major-seventh chords, but with the notes occurring in a slightly different order (see Figure 23).[63] Furthermore, if we reverse the order of two pairs of notes, as in the second system of the diagram, the resulting structure is strictly canonic, except for the location marked. (GS, of course, is not strictly canonic, and the parallel thirds shown in the second system of Figure 23 are not nearly as interesting as the counterpoint of Coltrane's composition.) It is also significant that Coltrane's addition of the ii chord to Slonimsky's sequential cell creates imitation within the cell itself (A-D). Since the cell recurs throughout the second

[62]My interpretation of the word "prolongation" here follows sketches and discussion by Demsey and Martin, and usage by Fred Lerdahl (not Heinrich Schenker). Demsey, "Chromatic Third Relations," 169ff.; Martin, "Expanding Jazz Tonality," 212ff; Fred Lerdahl, *Tonal Pitch Space* (New York: Oxford University Press, 2001), 350.
[63]However, recall that Chambers's bassline in the head features whole-tone descents.

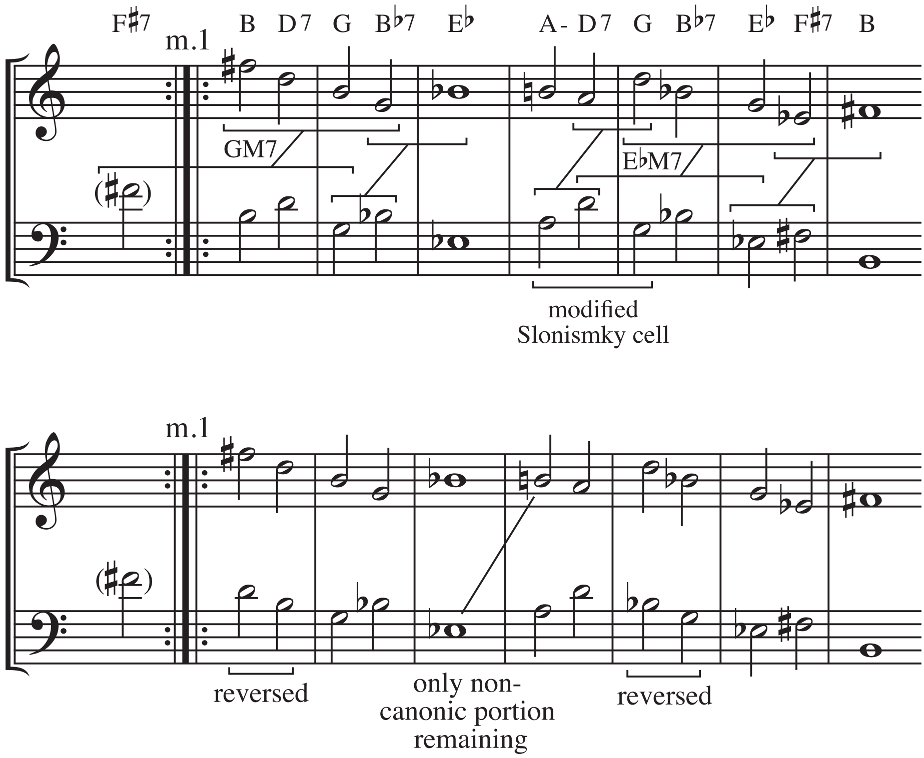


Figure 23. Melody and root relationships in the first half of "Giant Steps." All pitch-classes belong to Hex(2,3) except for the A's from the modified Slonimsky cell. In the second system the roots of measures 1 and 5 have been reversed to illustrate the quasi-canonic structure.

half of GS (though sometimes altered), partial imitation between melody and root tones is a unifying feature of the composition overall. This quasi-canonic structure sheds light on why pianist Tommy Flanagan stated, in reference to the sheet music he was given at the session, "I don't think there was any melody, just the chord sequence, which spells out the melody, practically."[64]

It is intriguing to consider the ways in which the Slonimsky cell may have shaped the composition overall. As discussed above, adding the ii chord creates imitation, a technique used throughout the piece. When the cell is sequenced, the added ii chord and Coltrane's omission of one of the melody tones creates the Southern nonatonic collection, which is also adhered to throughout the first half of the composition. The sequential design of the second half is reflected in the construction of the first half as well, which consists of two iterations of a four-bar unit that itself contains local-level harmonic and melodic sequences (see discussion of Figure 24, below). Moreover, Coltrane seamlessly interweaves the modified Slonimsky cell into the larger sequence of the first half (with the second iteration occurring as part of the overlapping

[64]Lewis Porter cites this quotation of Flanagan. Porter, *John Coltrane: His Life and Music* (Ann Arbor: University of Michigan Press, 1998), 155. Cedar Walton plays on the some of the alternate takes.

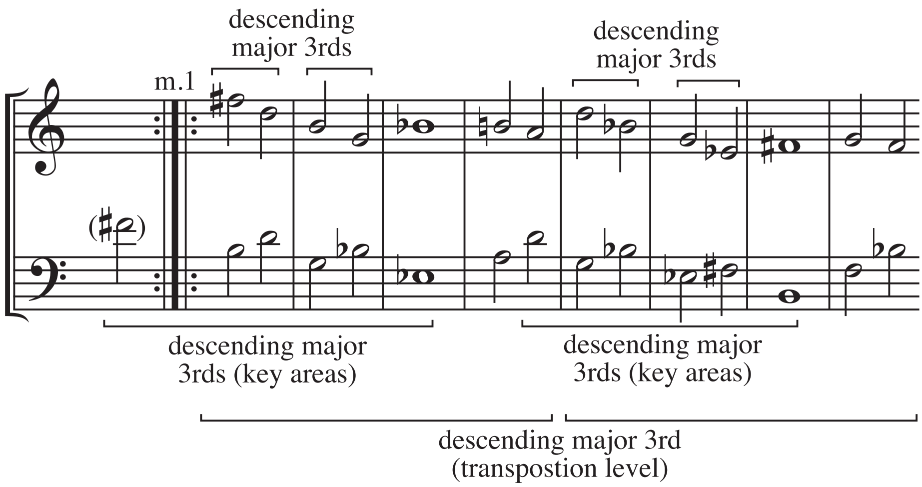


Figure 24. Motivic parallelism in the first half of "Giant Steps." The descending major-third motive appearing at three different structural levels.

measure discussed below). If the pattern in Figure 9 was the source of inspiration for GS, it would appear that Coltrane absorbed, extended, and exploited the full implications contained in that pattern to create a cohesive whole.

Finally, regarding the descending major-seventh chord arpeggiations, it must be observed that these deceptively simple falling thirds are a foreground manifestation of a descending major-thirds motive that exists on three different structural levels in the first half of GS, as shown in Figure 24. Two out of three of the descending thirds in the surface-level major-seventh arpeggios are major thirds, the key areas of mm. 16–3 and mm. 4–7 descend by major thirds, and mm. 5–8 are identical to mm. 1–4, transposed down a major third.[65]

"Giant Steps," "Steps of the Giant," and "Dual Duel"

As I have aimed to demonstrate throughout this article, there are many pieces to GS – nonatonic collections, twin ic4 cycles, melodic aspects, etc. – and they all fit together intricately and precisely. Given this puzzle-like construction, it is both natural and revealing to consider other ways in which the pieces of GS could fit together. As Yamaguchi notes:

> Coltrane's multi-tonic innovations imply further possibilities. My composition "Steps of the Giant" serves as an example of how these techniques may be extended.
>
> Compared to the structure of "Giant Steps" … "Steps of the Giant" can be seen as based on the opposite contour (descending/ascending) in chord changes (and melodies) … . We should not overlook that "Steps of the Giant" is still based on augmented triads … .[66]

[65] Measure 16 is the turnaround at the end of the GS form.

[66] Yamaguchi, "Multi-Tonic Changes," 160. Excerpts of "Steps of the Giant" are included by permission of Masaya Music Services.

Figure 25. The harmonic and melodic frameworks of Coltrane's "Giant Steps," Pellegrin's "Dual Duel," and Yamaguchi's "Steps of the Giant."

Yamaguchi does not identify the second half of "Steps of the Giant" as being based on the ic4a cycle, presumably because of the ii chords that intervene in the cycle. (This can be seen in his description and confirmed in his labeling of the composition itself.)[67] However, his composition does move through both ic4 cycles. Due to the "opposite contours" exhibited, as well as the title, this piece may be thought of as a sort of "dual" of GS, in keeping with the dualist preoccupation of Riemannian theory. In Figure 25 I build on this idea, presenting the harmonic and melodic framework of a composition that is aligned with GS and "Steps of the Giant" for comparison.

As indicated by the larger brackets beneath the scores, "Dual Duel" retrogrades the order of the large-scale prolongations in GS, from B–E♭ to E♭–B. In addition, the first three measures of both compositions prefigure these large-scale motions (see the smaller brackets beneath the scores). Both retrogrades are achieved by reversing the order of the ic4 cycles. However, since the order of the two halves is maintained, the ii chords are switched from the ic4b cycle to the ic4a cycle. As a result of this, the whole-tone bassline played by Chambers in the first half of GS appears in augmentation in the second half of "Dual Duel" (on the downbeats).

The melody of "Dual Duel" is essentially an inversion of the melody of GS. However, "Dual Duel" begins with an exact retrograde of the initial G major-seventh chord of GS, which effects a double interval-exchange with GS, as indicated by the brackets between the staves in mm. 1–2 of both pieces. The bracketed intervals 5955 in GS are then replaced with 3533 in "Dual Duel" as the first modified Slonimsky cell occurs; in the second half, these cells (again bracketed) consist of only three notes, and the 955 pattern is replaced with a 533 pattern. The melody has precisely the same range as GS: D♯/E♭^{4} to B♭^{5}. (With respect to GS this range is quite intriguing, since it has been argued that the piece is in E♭, and fifths above the root play a significant role in its construction, as discussed below.)[68]

Part of the modified Slonimsky cell – which occurs in mm. 4, 8, 10, 12, and 14 – did not invert well in "Dual-Duel"; the intervals were awkward and I did not want to resort to a common-tone option. I therefore transposed these portions of the cells up a major second. As in GS, substituting some common tones during these portions in the second half of the piece would smooth the transition to the repeated-note turnaround, and this is optional on repeats and in the outhead (also as in GS). However, I like this melodic sequence, in the way that melodic fragments from a single key ($\hat{6}\hat{7}\hat{3}\hat{4}\hat{5}\hat{1}$, indicated with dashed brackets) overlap key areas (as does the descending opening arpeggio of GS), and also overlap

[67]Ibid., 161.

[68]Nowadays, many theorists agree that the weight given to the final cadence of GS is significant, based upon Martin's theory of "prolongation by arrival," although this theory has been challenged by Waters. Martin, "Expanding Jazz Tonality"; Henry Martin and Keith Waters, "Hierarchy vs. Heterarchy in Two Compositions by Wayne Shorter," paper given at the annual meeting of the Society for Music Theory, Arlington, Virginia, 2017.

Figure 26. The two-bar hypermetric downbeats of “Giant Steps” and “Dual Duel.” In “Giant Steps,” these downbeats all consist of perfect fifths and therefore involve two different augmented triads. In “Dual Duel,” these downbeats are all major thirds and only use one augmented triad, creating a canon of hypermetric downbeats. Prolongations are also indicated, following Demsey. Hypermetric downbeats in the first half of “Giant Steps” mirror the surface-level root motions in the same section (illustrated in Figure 23).

the internal patterns themselves (as do the roots of the first half of GS, where two major-seventh chords – G and E♭ – overlap if we include the turnaround).[69]

The apex tones in the first half, F♯ and B♭/A♯ (the latter being the global-level apex tone), are repeated. However, since “Dual Duel” is a dual, it is really the lower apex tones that are significant. In addition, when the upper apex tones are repeated, they take the form of unresolved leading tones, which then recur sequentially in the second half of the piece (in measures 8, 10, 12, and 14). With the exception of the F♯ in the first half, all of these leading tones do resolve an octave lower though (several tones later), also in keeping with the dualistic nature of “Dual Duel.” While the key changes by the time the leading tones resolve, the melodic fragments themselves are diatonic – $\hat{6}\hat{7}\hat{3}\hat{4}\hat{5}\hat{1}$, as mentioned above. For example, the melody note D in m. 12 is the leading tone of E♭, the local tonic. By the time the D resolves to E♭ (D♯) in m. 15, via the E♭ major melodic fragment $\hat{6}\hat{7}\hat{3}\hat{4}\hat{5}\hat{1}$, the key has changed to B major.

The fact that “Dual Duel” works at all – that the various levels of harmonic and melodic structure may be manipulated while maintaining coherence and

[69]The overlapping major-seventh chords are easier to see in Figure 23, where the dominant chord of the turnaround is shown.

the order of the sections – further indicates the highly organized and dualistic nature of GS. At the same time, comparing my dual with GS uncovers additional reasons why the latter is an exceptional composition. In general, my piece is simpler than GS. For example, my piece is not canonic the way GS is (but see below). Other aspects require more explanation.

I mentioned harmonic prolongation above, but melodic prolongation is present as well.[70] As shown in Figure 26, the first half of GS begins and ends with F♯ over B and the second half begins and ends with B♭ over E♭. The first half of "Dual Duel" begins and ends with G over E♭ and the second half begins and ends with D♯ over B. In addition, a perfect fifth occurs on every two-bar hypermetric downbeat in GS, whereas a major third occurs on every two-bar hypermetric downbeat in my piece.[71] But these thirds in my composition tip the overall balance too far in the direction of major thirds. (The major thirds are balanced somewhat in that every hypermetric upbeat, except for the final measure, is a perfect fifth.) For example, in the second half, as the sequence descends through a major-third cycle, the melody trails the chord roots on these hypermetric downbeats, as it cycles not only through an augmented triad (as in GS) but through the same augmented triad. In fact, in the first half of "Dual Duel," the melody on every two-bar hypermetric downbeat also echoes the roots, creating a canon of hypermetric downbeats running through the piece. However, this hypermetric canon, rather than creating complexity, creates redundancy and empty simplicity (especially in the second half), merely repeating the major-thirds cycle already present. Coupled with the quality of the major thirds, the hypermetric canon makes my piece less complex, less bold, and less interesting than GS. In GS, the melody notes a fifth above the roots outline a second augmented triad, completing the $\text{Hex}_{(2,3)}$ collection. The fifths of GS also complement the major thirds occurring elsewhere, adding depth, openness, aggressiveness, and richness to the sound. Lastly, Figure 26 reveals that the hypermetric downbeats in the first half of GS mirror the surface-level root motions in the same section (illustrated in Figure 23), but outline an augmented triad rather than a major-seventh chord.[72]

There is another way in which "Dual Duel" is too simple with respect to the relationship between melody and harmony. Recall that in GS, all of the melody notes and chord roots belong to the Southern system, whereas all of the zero- or quasi zero-sum realizations were drawn from the Western system. In "Dual Duel," all of the melody notes are drawn from the Western system as well. The complete set of chord roots are still drawn from the Southern system, due to the ii chords, but the ii chords are already omitted in all of the zero-sum realizations. Without the ii chords (in considering both chord roots and harmonic realizations), both the melody and the zero-sum harmonic realizations of "Dual Duel" fit into

[70]See Demsey's sketch. Demsey, "Chromatic Third Relations," 170.
[71]These downbeats are supported not only by hypermetric regularity but also by either duration – whole notes – or a full ii-V-I progression.
[72]This is true regardless of which octave the roots are written in.

the same nonatonic system. Thus, it can again be observed that there is a closer relationship between melody and harmony in my piece, giving it a simpler sound, to the extent that it lacks the interest of GS.[73] Since it takes two nonatonic collections to complete the aggregate, "Dual Duel" is less chromatic overall, and the Western and Southern nonatonic collections in GS complement one another.

Conclusion

For all of the seamlessness created by the zero-sum characteristics of GS, and separately, its remarkable hexatonic and nonatonic properties, these two domains overlap but do not map precisely onto one another, creating richness and complexity. Similarly, while there are clear patterns in GS, there are also several instances of quasi-patterns, or other aspects of design that create comparatively rough edges. For example, the melody and roots of the first half are *almost* canonic, the melodic pattern in the second half is often altered with repeated tones, the phrasal and hypermetric boundaries overlap, the fifths on hypermetric downbeats (in the first half) move through a complete hexatonic collection rather than cycling canonically but redundantly through one augmented triad (as in "Dual Duel"), and so forth. Other patterns in GS lie under the surface, such as the major-thirds motivic parallelism (Figure 24) and the parallelism between the hypermetric downbeats and the surface-level root motions (both in the first half).

GS is a tightly-woven, theoretically-dense composition, so much so that Coltrane wrote in the original liner notes, "I'm worried that sometimes what I'm doing sounds just like academic exercises." However, the theoretical neatness of GS is offset by several irregularities, creating a cohesive and nuanced whole which – rather than being an academic exercise like the pattern from Slonimsky's introduction or my foil – is one of the most enduring and significant jazz compositions.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes on contributor

Rich Pellegrin is Assistant Professor of Music Theory at the University of Florida. His dissertation from the University of Washington, "On Jazz Analysis: Schenker, Salzer, and Salience," examined the significance of the Salzerian analytical tradition with respect to both the classical and jazz idioms. He has presented his research at numerous regional, national, and international conferences. His work has been published in *Engaging Students: Essays in*

[73]To clarify, three factors were mentioned that indicate a closer relationship between melody and harmony in "Dual Duel": (1) the melody and harmony are drawn from the same nonatonic collection, (2) there is a hypermetric canon between the chord roots and the melody, and (3) the chords roots and the melody notes in the hypermetric canon are drawn from the same augmented triad rather than two different ones.

Music Pedagogy, the *Journal of Schenkerian Studies*, and in volumes by Cambridge Scholars Publishing and KFU Publishing House. He is currently working on a book entitled *Analytical Approaches to Jazz: Tonal, Modal, and Beyond*. Active as a jazz pianist and composer, Dr. Pellegrin's fourth album for Origin Records' OA2 label will be released in 2021.

Appendices

Appendix 1. Nonatonic Collections and Non-Mutual Exclusivity

As mentioned in the main text, the calculations presented in this article – particularly in appendix 2 – rule out the possibility of intersecting events and therefore the issue of non-mutual exclusivity does not arise. The purpose of this appendix is to demonstrate to interested readers how one would address non-mutual exclusivity if the problem were approached in such a way that it was necessary to do so. I will address this question in the context of the following equation from the main text of the article:[74]

$$0.075\% + 0.075\% + 0.075\% + 0.075\% = 0.3\%$$

The four nonatonic collections are non-mutually exclusive. The 0.075% (rounded) probability for each collection intersects with the probability for the other collections; these intersecting probabilities must be subtracted in order to avoid counting them more than once.

The steps below demonstrate how to incorporate the non-mutually exclusive properties of nonatonic collections into the general calculation of probability. In Step One, the probability for each collection is added together (as in the reference equation from the main text, shown above).

Step One:

$$\begin{aligned} &P(N) + P(E) + P(S) + P(W) \\ &= (9/12)^{25} + (9/12)^{25} + (9/12)^{25} + (9/12)^{25} \\ &= 4(9/12)^{25} \end{aligned}$$

In Step Two, the probability for the intersection of each group of *two* collections is *subtracted*, because it was counted too many times in Step One. Each pair of collections contains six overlapping pitch classes, as illustrated in Figure 5.

Step Two:

$$\begin{aligned} &-P(N \cap E) - P(E \cap S) - P(S \cap W) - P(W \cap N) - P(N \cap S) - P(E \cap W) \\ &= -(6/12)^{25} - (6/12)^{25} - (6/12)^{25} - (6/12)^{25} - (6/12)^{25} - (6/12)^{25} \\ &= -6(6/12)^{25} \end{aligned}$$

In Step Three, the probability for the intersection of each group of *three* collections is *added*, because it was *subtracted* too many times in Step Two. Every group of three collections contains three overlapping pitch classes; again, see Figure 5.

Step Three:

$$\begin{aligned} &P(N \cap E \cap S) + P(E \cap S \cap W) + P(S \cap W \cap N) + P(W \cap N \cap E) \\ &= (3/12)^{25} + (3/12)^{25} + (3/12)^{25} + (3/12)^{25} \\ &= 4(3/12)^{25} \end{aligned}$$

[74] NB: The discussion in this appendix applies *only* to this equation and its meaning as presented in the main text. Considerations which were introduced after this point in the main text may not apply in other approaches to the problem, and are not addressed here.

In Step Four, the probability for the intersection of each group of *four* collections is *subtracted*, because it was *added* too many times in Step Three. There is only one group of four collections, and it contains zero overlapping pitch classes (see Figure 5).

Step Four:

$$-P(N \cap E \cap S \cap W)$$
$$= -(0/12)^{25}$$

The preceding may be summarized as follows:

Step One: $P(N) + P(E) + P(S) + P(W)$
Step Two: $-P(N \cap E) - P(E \cap S) - P(S \cap W) - P(W \cap N) - P(N \cap S) - P(E \cap W)$
Step Three: $P(N \cap E \cap S) + P(E \cap S \cap W) + P(S \cap W \cap N) + P(W \cap N \cap E)$
Step Four: $-P(N \cap E \cap S \cap W)$

Numerically, the steps may be summarized like so:

Step One: $(9/12)^{25} + (9/12)^{25} + (9/12)^{25} + (9/12)^{25}$
Step Two: $-(6/12)^{25} - (6/12)^{25} - (6/12)^{25} - (6/12)^{25} - (6/12)^{25} - (6/12)^{25}$
Step Three: $(3/12)^{25} + (3/12)^{25} + (3/12)^{25} + (3/12)^{25}$
Step Four: $-(0/12)^{25}$

These three steps may then be written more succinctly, combined into one equation, and evaluated:

$$4(9/12)^{25} - 6(6/12)^{25} + 4(3/12)^{25} - (0/12)^{25} = 0.003009995 = 0.3009995\%$$

It might be observed that this result is greater than the 0.3% given in the reference equation. We would expect this value to be smaller, reflecting the fact that redundancies have been subtracted. The reason for this is that the reference equation in the main text was provided with rounded values in order to demonstrate how a potential decimal error could have yielded the amount of "less than 3%" given by Santa. The reference equation uses rounded values derived from the following:

$$(9/12)^{25} + (9/12)^{25} + (9/12)^{25} + (9/12)^{25}$$

If more decimal places are provided, we arrive at:

$$0.003010174 = 0.3010174\%$$

This value, 0.3010174%, is greater than the 0.3009995% given above, which makes sense since the latter amount was arrived at by subtracting redundancies.

Appendix 2. Eliminating Subsets: A More Precise Calculation

The purpose of this appendix is to eliminate the possibility of subsets more precisely than the calculation given in the main text. Before doing so, I will provide an explanation of the issue of subsets, and of how the calculations in this appendix differ from those given in Appendix 1.

9/12 expresses the likelihood that a randomly-selected pitch-class belongs to a given nine-note collection. $(9/12)^n$ indicates that *n* independent, random selections are performed. There are no stipulations regarding the end result of these trials other than the fact that every pitch-class must be a member of the nonatonic collection. There is thus the possibility that the result forms a subset of the target collection that is also a subset of other nonatonic collections. For example, the Northern hexatonic collection is a subset of both the Northern and Western nonatonic collections (see Figure 8). If we do not eliminate the possibility of subsets, then there is a chance that the randomly-selected pitch-classes could form the Northern hexatonic collection.

Suppose the target collection was the Northern nonatonic collection but the random selections resulted in the Northern hexatonic collection. In such a case, the conditions for P(N) would be met, as all members of the set belong to the Northern nonatonic collection; however, the conditions for P(E) would also be met. The events N and E intersect due to the possibility of subsets being formed.

In Appendix 1, the probabilities for each of the four nonatonic collections were first added together in order to determine the likelihood that any one of them resulted. But if the probability for P(N) and P(E) were merely added together, then the probability for the Northern hexatonic collection would be counted twice – once as part of P(N) and once as part of P(E). We must therefore subtract this probability once in a later step, resulting in the process described.

In the present appendix, the calculations eliminate the possibility that subsets of nonatonic collections are formed. We are calculating the probability that a collection is *exactly* nonatonic – that it contains no fewer and no greater pitch-classes than those in a nonatonic collection. The resulting nine-notes can only belong to one nonatonic collection. (*Some* of the notes of the collection – i.e. subsets – will belong to other nonatonic collections, but the complete set of nine-notes may only belong to one nonatonic collection.)

As mentioned in the main text, the following calculation is a simplification:

$$(9/12)^{25} - (8/12)^{25} = 0.00071294133 = 0.071\%$$

Our ultimate goal is to understand this problem when both the melody tones and the chord roots are considered, so our reference equation will be the next expression occurring in the main text:

$$(9/12)^{51} - (8/12)^{51} = 0.00000042369 = 0.00004\%$$

Below, this calculation is performed more precisely. With such a large exponent, only a slight difference results from the process undertaken here, and therefore the main text is essentially unaffected.

$(9/12)^{51}$ is the probability that fifty-one pitch classes, selected randomly in independent trials, forms the Southern nonatonic collection or a subset thereof. When we subtract $(8/12)^{51}$, we are subtracting the possibility of various subsets of the Southern collection being formed; this includes eight-note subsets, seven-note subsets, six-note subsets, and so forth. However, some things are subtracted too many times in the process; thus, refinements to the calculation must be made.[75]

As with $(9/12)^{51}$, the subtracted value of $(8/12)^{51}$ includes the probability of subsets. For example, it includes seven-note subsets. But some of these seven-note subsets will be identical, have therefore been subtracted out too many times, and must be added back. Suppose there is a nine-note collection {012345678}. Consider the two eight-note subsets {01234567} and {12345678}. The seven-note collection {1234567} is a subset of both eight-note collections. If the probability of {1234567} is subtracted once for {01234567} and once for {12345678}, then it must be added back once in a later step. However, when we add back the probability of certain seven-note subsets, we are *adding* some *six*-note

[75]In this context, $9(8/12)^{51}$ is more accurate than $(8/12)^{51}$, but the process which results in multiplying by nine is not described until further below.

subsets too many times, and therefore must *subtract* them back. We then subtract some five-note subsets too many times, and so forth.

For the sake of explanation, I will demonstrate the construction of the equation given at the end of this appendix by breaking down its components. The basic structure of the left side of the equation may be considered roughly as:

$$\text{P(9-note collection)} - \text{P(8-note subsets)} + \text{P(7-note subsets)} - \text{P(6-note subsets)} \ldots + \text{P(1-note subsets)}$$

The terms of this expression, in simplified form, are as follows:

$$\left(\frac{9}{12}\right)^{51} - \left(\frac{8}{12}\right)^{51} + \left(\frac{7}{12}\right)^{51} - \left(\frac{6}{12}\right)^{51} \ldots + \left(\frac{1}{12}\right)^{51}$$

This may be rewritten as such:

$$\frac{9^{51} - 8^{51} + 7^{51} - 6^{51} \ldots + 1^{51}}{12^{51}}$$

We must now include calculations that take into consideration the number of possible ways that each collection may be counted.[76] In this next step, one new element is incorporated within each successive term of the expression: 9/1, followed by 8/2, followed by 7/3, and so forth.[77]

$$\frac{9^{51} - \frac{9}{1}8^{51} + \frac{9\bullet 8}{1\bullet 2}7^{51} - \frac{9\bullet 8\bullet 7}{1\bullet 2\bullet 3}6^{51} \ldots + \frac{9\bullet 8\bullet 7\bullet 6\bullet 5\bullet 4\bullet 3\bullet 2}{1\bullet 2\bullet 3\bullet 4\bullet 5\bullet 6\bullet 7\bullet 8}1^{51}}{12^{51}}$$

There are nine unique eight-note subsets of the nonatonic collection – there are nine ways of ignoring one note – and each of these eight-note subsets occurs once. This is expressed with 9/1. For each of these eight-note subsets there are eight unique seven-note subsets. Each seven-note subset occurs twice, and thus we divide by two, arriving at 8/2. (9 • 8) / (1 • 2) expresses the number of unique seven-note subsets that need to be added back after being subtracted too many times with the eight-note subsets. The process continues in similar fashion until one-note subsets have been considered.

The expression can now be re-notated and evaluated.

$$\frac{9^{51} - 9\bullet 8^{51} + \frac{9\bullet 8}{2}7^{51} - \frac{9\bullet 8\bullet 7}{3\bullet 2}6^{51} \ldots + \frac{9\bullet 8\bullet 7\bullet 6\bullet 5\bullet 4\bullet 3\bullet 2}{8\bullet 7\bullet 6\bullet 5\bullet 4\bullet 3\bullet 2}1^{51}}{12^{51}} = 0.00000041$$

This value is slightly lower than the 0.00000042 arrived at in the main text. Finally, we multiply by four, since the four nonatonic collections are transpositionally equivalent.

$$4\left(\frac{9^{51} - 9\bullet 8^{51} + \frac{9\bullet 8}{2}7^{51} - \frac{9\bullet 8\bullet 7}{3\bullet 2}6^{51} \ldots + \frac{9\bullet 8\bullet 7\bullet 6\bullet 5\bullet 4\bullet 3\bullet 2}{8\bullet 7\bullet 6\bullet 5\bullet 4\bullet 3\bullet 2}1^{51}}{12^{51}}\right) = 0.00000166$$

[76] There are a fixed set of collections, but a number of ways to count them. For any given approach to this calculation, it must necessarily be the case that each collection is ultimately counted once and only once.

[77] Numerals are normally written in descending order, with the 1s omitted, to facilitate solving; I have written them in ascending order here, with the 1s included, to facilitate understanding of the construction itself.